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On Construction of Rhotrix Semigroup

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Abstract

The concept of rhomboidal arrays,now known as rhotrices, was introduced in 2003 as a new paradigm of matrix theory of rectangular arrays. This paper presents a construction of certain algebraic system, which we termed as rhotrix semigroup. We identify the properties of this rhotrix semigroup and characterize its Green's relations. Furthermore, as comparable to regular semigroup of square matrices, we show that the rhotrix semigroup is also a regular semigroup.

Key words: Rhotrix, Matrix, Semigroup, Rhotrix semigroup**,** Matrix semigroup.

1.0 Introduction

Ever since thebirth of the concept of rhotrix by Ajibade[1], as an extension of ideas on matrix-tertions and matrix-noitrets proposed by Atanassov and Shannon[2], there have been much interest by some Authors, in the usage of rhotrix set as an underlying set, in construction of algebraic structures (see, Mohammed[3], Mohammed [4],Tudunkaya and Makanjuola [5], Usaini and Tudunkaya [6] and Usaini and Tudunkaya [7]).

Nevertheless, it is noteworthy to mention that all the rhotrix algebraic systems considered by all the above Authors are based on either one or two of rhotrix operations defined in Ajibade [1]. So this motivated us to draw our attention next to Sani [8], whose paper following the lead author, proposed an alternative method for multiplication of base rhotrices (also known rhotrices of size-3), in an attempt to answer the question of "finding a transformation for conversion of rhotrix to matrix and vice versa", in the concluding section of Ajibade's article. This multiplication proposed by Sani[8] was thereafter, extended to higher dimensional rhotrices, in form of generalization by Sani[8].To the best of our knowledge, construction of semigroup using rhotrix set as an underlying set, together with the binary operation of rhotrix multiplication proposed by Sani [9],has never been done in the literature of rhotrix theory.

Thus, this paper presents a novel method of constructing certain semigroup using rhotrix set as the underlying set, together with the binary operation for rhotrix multiplication proposed by Sani [9].An identification of some of its properties and characterization of its Green's relations will be presented. Analogously to regular semigroup of square matrices, we shall show that the rhotrix semigroupis also a regular semigroup.

Now, we start with the following preliminary section, which highlight fundamental ideas that may help in our discussion of subsequent sections.

2.0 Preliminaries

2.1 Rhotrix Set

A rhotrix set is a set consisting of rhotrices of the same size with entries from a fixed field.

For example, arhotrix set consisting of all rhotrices of size *3*over the set of real numbers is given by

$$R_3(\Re) = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in \Re \right\}. \qquad (1)$$

Also, arhotrix set consisting of all rhotrices of size*n*, $(n \in 2Z^{+} + 1)$ over an arbitrary field *F* is given by

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$$R_n(F)=\left\{\left\langle\begin{array}{ccccccccccc} & & & & & a_{11} & & & & & \\ & & & & a_{21} & c_{11} & a_{12} & & & & \\ & & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_{tt-2} & c_{t-1t-2} & a_{t-1t-1} & c_{t-2t-1} & a_{t-2t} & & & \\ & & & & a_{tt-1} & c_{t-1t-1} & a_{t-1t} & & & & \\ & & & & & a_{tt} & & & & & \end{array}\right\rangle : a_{ij}, c_{kl}\in F\right\}, \qquad (2)$$

where a_{ij} and c_{lk} represent respectively, the a_{ij} and c_{lk} elements from arbitrary field *F*, with $i, j = 1,2,3......,t$ and $l,k = 1,2,3,.......,t-1.$ Also, $t = (n+1)/2$ and $n \in 2Z^{+} + 1.$

2.2 Rhotrix Multiplication

In the literature of Rhotrix Theory, two different methods are available for multiplication of rhotrices having the same size. The first one was given in the origin of rhotrix concept by Ajibade [1], recorded as follows:

Let *A* and *B* be base rhotrices then their product is defined by

$$A\circ B=\left\langle\begin{array}{ccc} & a_1 & \\ a_2 & h(A) & a_4 \\ & a_5 & \end{array}\right\rangle\circ\left\langle\begin{array}{ccc} & b_1 & \\ b_2 & h(B) & b_4 \\ & b_5 & \end{array}\right\rangle=\left\langle\begin{array}{ccc} & a_1h(B)+b_1h(A) & \\ a_2h(B)+b_2h(A) & h(A)h(B) & a_4h(B)+b_4h(A) \\ & a_5h(B)+b_5h(A) & \end{array}\right\rangle, \qquad (3)$$

where $a_3 = h(A)$ is called heart of rhotrix *A*and also, $b_3 = h(B)$ is called heart of rhotrix *B*. This product was later given a generalisation in Mohammed [10], recorded as follows:

Let *A* and *B* be any two rhotrices of the same size*n* then their product,*A o B*is the resultant rhotrix *C* defined as

$$C(n)=A(n)\circ B(n)=\left\langle\begin{array}{ccccccccccc} & & & & & a_1 & & & & & \\ & & & & a_2 & a_3 & a_4 & & & & \\ & & & : & : & : & : & : & & & \\ & & : & : & : & : & : & : & : & & \\ a_{\left\{\frac{t+1}{2}\right\}-n\backslash 2} & .. & .. & .. & & a_{\left\{\frac{t+1}{2}\right\}} & & .. & .. & .. & r_{\left\{\frac{t+1}{2}\right\}+n\backslash 2} \\ & & : & : & : & : & : & : & : & & \\ & & & : & : & : & : & : & & & \\ & & & & r_{t-3} & r_{t-2} & r_{t-1} & & & & \\ & & & & & r_t & & & & & \end{array}\right\rangle\circ\left\langle\begin{array}{ccccccccccc} & & & & & b_1 & & & & & \\ & & & & b_2 & b_3 & b_4 & & & & \\ & & & : & : & : & : & : & & & \\ & & : & : & : & : & : & : & : & & \\ b_{\left\{\frac{t+1}{2}\right\}-n\backslash 2} & .. & .. & .. & & b_{\left\{\frac{t+1}{2}\right\}} & & .. & .. & .. & b_{\left\{\frac{t+1}{2}\right\}+n\backslash 2} \\ & & : & : & : & : & : & : & : & & \\ & & & : & : & : & : & : & & & \\ & & & & b_{t-3} & b_{t-2} & b_{t-1} & & & & \\ & & & & & b_t & & & & & \end{array}\right\rangle$$

$$=\left\langle\begin{array}{ccccccccc} & & & & a_1h(B)+b_1h(A) & & & & \\ & & & r_2h(B)+b_2h(A) & r_3h(B)+b_3h(A) & r_4h(B)+b_4h(A) & & & \\ & & : & : & : & : & : & & \\ & : & : & : & : & : & : & : & \\ a_{\left\{\frac{t+1}{2}\right\}-n\backslash 2}h(B)+b_{\left\{\frac{t+1}{2}\right\}-n\backslash 2}h(A) & .. & .. & .. & h(A)h(B) & .. & .. & .. & a_{\left\{\frac{t+1}{2}\right\}+n\backslash 2}h(B)+b_{\left\{\frac{t+1}{2}\right\}+n\backslash 2}h(A) \\ & : & : & : & : & : & : & : & \\ & & : & : & : & : & : & & \\ & & & a_{t-3}h(B)+b_{t-3}h(A) & a_{t-2}h(B)+a_{t-2}h(A) & a_{t-1}h(B)+b_{t-1}h(A) & & & \\ & & & & a_th(B)+b_th(A) & & & & \end{array}\right\rangle \qquad (4)$$

where $n\in 2Z^{+}+1$, $t=\frac{1}{2}(n^2+1)$, $h(A)=a_{\left\{\frac{t+1}{2}\right\}}$, $h(B)=b_{\left\{\frac{t+1}{2}\right\}}$, $h(C)=c_{\left\{\frac{t+1}{2}\right\}}$ and $n\backslash 2$ is the integer value obtained on division of *n* by *2*.

However, an alternative method for multiplication of base rhotrices, using row and column approach, as comparable to matrices was proposed by Sani [8], which was later given a generalization in Sani [9], recorded as follows:

For any $R, Q \in R_n(F)$, the rhotrix multiplication of *R* and *Q* is

$$R_n \circ Q_n = \langle a_{i1j1}, c_{l1k1} \rangle \circ \langle b_{i2j2}, d_{l2k2} \rangle = \left\langle \sum_{i2j1=1}^{t} \left(a_{i1j1} b_{i2j2} \right), \sum_{l2k1=1}^{t-1} \left(c_{l1k1}, d_{l2k2} \right) \right\rangle \quad (5)$$

where,

$$R_n = \langle a_{ij}, c_{kl} \rangle = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & a_{1t} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & a_{tt-2} & c_{t-1t-2} & a_{t-1t-1} & c_{t-2t-1} & a_{t-2t} & & \\ & & & a_{tt-1} & c_{t-1t-1} & a_{t-1t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle \quad (6)$$

and

$$Q_n = \langle b_{ij}, d_{kl} \rangle = \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & b_{21} & d_{11} & b_{12} & & & \\ & & b_{31} & d_{21} & b_{22} & d_{12} & b_{13} & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ b_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & b_{1t} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & b_{tt-2} & d_{t-1t-2} & b_{t-1t-1} & d_{t-2t-1} & b_{t-2t} & & \\ & & & b_{tt-1} & d_{t-1t-1} & b_{t-1t} & & & \\ & & & & b_{tt} & & & & \end{array} \right\rangle \quad (7)$$

such that, a_{ij} and c_{lk} represent the a_{ij} and c_{lk} elements respectively, with $i, j = 1,2,3......,t$ and $l, k = 1,2,3,.......,t-1.$ And where $t = (n+1)/2$ and $n \in 2Z^+ + 1.$

Also, b_{ij} and d_{lk} represent the b_{ij} and d_{lk} elements respectively, with $i, j = 1,2,3......,t$ and $l, k = 1,2,3,.......,t-1.$ And where $t = (n+1)/2$ and $n \in 2Z^+ + 1.$
For details, regarding identity, inverse, transpose and determinant with respect to the row-column method of rhotrix multiplication, see Sani [9].

It observed by Sani [11] that, if $t = \frac{(n+1)}{2}$ and $n \in 2Z^+ + 1$ then every $n \times n$ rhotrix, denoted by R_n, may be represented as a couple of two matrices given by $R_n = \langle a_{ij}, c_{kl} \rangle,$ where (a_{ij}) is a $t \times t$ matrix (called the major matrix of R_n) and (c_{kl}) is a $(t-1) \times (t-1)$ matrix (called the minor matrix of R_n).

3.0 The Rhotrix Semigroup

Let $R_n(F)$ be a set of all rhotrices of the same size *n*over a field *F*,together with rhotrix operation of addition or rhotrix operation of multiplication then $\langle R_n(F), + \rangle$ or $\langle R_n(F), \circ \rangle$ is called respectively as, rhotrix semigroup over addition or rhotrix semigroup over multiplication.

Remark

a. If $R_n(F)$ is a rhotrix semigroup with rhotrix operation of addition then $\langle R_n(F), + \rangle$ is a commutative semigroup of all rhotrices of the same size *n*.

b. If $R_n(F)$ is a rhotrix semigroup with rhotrix operation of multiplication defined by Ajibade [1] then $\langle R_n(F),\circ\rangle$ is a commutative semigroup of all rhotrices of the same size *n*. This semigroup was already considered by Mohammed [3].

c. If $R_n(F)$ is a rhotrix semigroup with rhotrix operation of multiplication defined by Sani [9] then $\langle R_n(F),\circ\rangle$ is a non-commutative semigroup of all rhotrices of the same size *n*.

In this article, we shall study the semigroup indicated in our remark c. The semigroup $\langle R_n(F),\circ\rangle$, with respect to Sani's methodfor multiplication of rhotrices forms an interesting algebraic semigroup because of its non-commutative structure. This motivated us to consider its study,by asking several questions such as follows:

i. What type of semigroup is $\langle R_n(F),\circ\rangle$?
ii. What are its Green's relations like?
iii. What are its idempotent elements?
iv. What are its nilpotent elements?
v. What are its subsemigroups?
vi. Can we construct a mapping between its subsemigroups, such that they are homomorphic?
vi What type of homomorphism exist between its subsemigroups?
vii Is there any link between rhotrix semigroup and matrix semigroup?
and *etc*.

We shall attempt to answer all these questions in our subsequent discussions. Furthermore, throughout the remaining sections of this paper, we shall mean $\langle R_n(F),\circ\rangle$ to be a semigroup with respect to Sani's multiplication.

3.1 Basic properties of rhotrix semigroup

Let $\langle R_n(F),\circ\rangle$ be a rhotrix semigroup $\langle R_n(F),\circ\rangle$ with respect to Sani's rhotrix multiplication, then $\langle R_n(F),\circ\rangle$ possesses the following basic properties:

(a) Non-Commutativity: For all rhotrices $A, B \in \langle R_n(F),\circ\rangle$, we have $A \circ B \neq B \circ A$

(b) Existence of identity element: The rhotrix semigroup $\langle R_n(F),\circ\rangle$ has an identity element *I*, as a unity element, such that $\forall A \in \langle R_n(F),\circ\rangle$, we have $A \circ I = I \circ A = A$. Hence, $\langle R_n(F),\circ\rangle$ is a monoid semigroup.

(c) Existence of zero or neutral element: The rhotrix semigroup $\langle R_n(F),\circ\rangle$ has a zero element *O*, as a neutral element, such that $\forall A \in \langle R_n(F),\circ\rangle$, we have $A \circ O = O \circ A = O$. Hence, $\langle R_n(F),\circ\rangle$ is a semigroup with zero.

(d) Infiniteness: The rhotrix semigroup $\langle R_n(F),\circ\rangle$ has unlimited number of elements if *F* is an infinite field. Otherwise, it is finite

3.2 Theorem

The rhotrix semigroup $\langle R_n(F),\circ\rangle$ is embedded in the matrix semigroup $\langle M_n(F),\cdot\rangle$, with respect to usual matrix multiplication.

Proof:

Let $\langle R_n(F),\circ\rangle$ be a rhotrix semigroup and let $M_n(F)$ be a matrix semigroup,with respect to matrix multiplication. We define a mapping

$$\theta : \langle R_n(F),\circ\rangle \to \langle M_n(F),\cdot\rangle$$

By

$$\theta\left(\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{tt-2} & c_{t-1t-2} & a_{t-1t-1} & c_{t-2t-1} & a_{t-2t} & & \\ & & & a_{tt-1} & c_{t-1t-1} & a_{t-1t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle\right) = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1t-1} & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{2t} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ a_{t-11} & 0 & a_{t-12} & 0 & \cdots & \cdots & \cdots & 0 & a_{t-1t} \\ 0 & c_{t-11} & 0 & c_{t-12} & \cdots & \cdots & \cdots & c_{t-1t-1} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt} \end{pmatrix} \quad (8)$$

Clearly, it is simple to verify that θ is an injective homomorphism.

3.3 Theorem

The semigroup $\langle R_n(F), \circ \rangle$ is a regular semigroup.

Proof

We are to show that for each rhotrix $A \in R_n(F),$ there exist a rhotrix $B \in R_n(F)$ such that $A \circ B \circ A = A$.

Now, let rhotrix $A = \langle a_{ij}, c_{kl} \rangle \in R_n(F)$. Since the semigroup of all square matrices over F is regular, then, the major matrix (a_{ij}) and the minor matrix (c_{kl}) are regular elements of $M_t(F)$ and $M_{t-1}(F)$ respectively. Thus, there exist a matrix $(b_{ij}) \in M_t(F)$ and a matrix $(d_{kl}) \in M_{t-1}(F)$ such that $(\mathrm{a_{ij}})(\mathrm{b_{ij}})(\mathrm{a_{ij}}) = (\mathrm{a_{ij}})$ and $(\mathrm{c_{kl}})(\mathrm{b_{kl}})(\mathrm{c_{kl}}) = (\mathrm{c_{kl}})$ respectively. Now, choose a rhotrix $B = \langle \mathrm{b_{ij}}, \mathrm{d_{kl}} \rangle \in R_n(F)$. Then, it follows from the definition of rhotrix multiplication that $A \circ B \circ A = A$. Hence the result.

4.0 Green's relations in $\langle R_n(F), \circ \rangle$

It is customary that when one encounters a new class of semigroup, the first question to ask, is what are the Green'srelations like?Therefore, as a first step in understanding the structure of our new semigroup, $\langle R_n(F), \circ \rangle$, we present in this section, a characterization of Green's relations in $\langle R_n(F), \circ \rangle$.

Recall that, in a semigroup S , we define Green's equivalences $L, \Re, J, H$ and D as

aLb if and only if $(\exists x, y \in S^1) a = xb$ and $b = ya$;

$a\Re b$ if and only if $(\exists u, v \in S^1) a = bu$ and $b = av$;

aJb if and only if $(\exists x, y, u, v \in S^1) a = xby$ and $b = uav$;

cDb if and only if $(\exists c \in S)$ aLc and $c\Re b$; and

$H = L \cap \Re$.

First, we make the following observation concerning the semigroup $\langle R_n(F), \circ \rangle$ and Green's equivalences.

4.1 Lemma

Let κ denote any of the five Green's equivalencesL, R, J, H and D then for any $A = \langle a_{ij}, c_{kl} \rangle$ and $B = \langle a_{ij}, c_{kl} \rangle$ in $\langle R_n(F), \circ \rangle$, we have:

$A\kappa B$ if and only if $(a_{ij})\kappa(b_{ij})$ and $(c_{kl})\kappa(d_{kl})$.

Proof
We prove the result for only the Green's equivalence Land the proof for R,J, Hand D follows similarly.
Let$A$L $B \Leftrightarrow \exists X, Y \ni A = X \circ B$ and $B = Y \circ A$
Let

$$\Leftrightarrow \quad \langle a_{ij}, c_{kl} \rangle = \langle (x^1{}_{ij})(b_{ij}), (x^2{}_{kl})(d_{kl}) \rangle \ \ and \ \ \langle b_{ij}, d_{kl} \rangle = \langle (y^1{}_{ij})(a_{ij}), (y^2{}_{kl})(c_{kl}) \rangle$$

$$\Leftrightarrow \quad a_{ij} = (x^1{}_{ij})(b_{ij}), c_{kl} = (x^2{}_{kl})(d_{kl}) \ \ and \ \ b_{ij} = (y^1{}_{ij})(a_{ij}), d_{kl} = (y^2{}_{kl})(c_{kl})$$

$$\Leftrightarrow \quad (a_{ij})L(b_{ij}) \ \ and \ \ (c_{kl})L(d_{kl}).$$

This completes the proof.

To characterize Green's relations on $\langle R_n(F), \circ \rangle$, we record the following result from Howie [12]

4.2 Theorem [Howie[12], Proposition 2.4.2]

Let$U$ be a regular subsemigroup of a semigroup S. Then we have

(i) L (U)=L (S) $\cap (U \times U)$,

(ii) R (U)=R (S) $\cap (U \times U)$, and

(iii) H (U)=H (S) $\cap (U \times U)$,

Thus, the above theorem shows that any regular subsemigroup U of a semigroup S will have the same characterization of Green's relations L, R, and H as in S. Now, since the semigroup $\langle R_n(F), \circ \rangle$ is a regular subsemigroup of $\langle M_n(F), \cdot \rangle$, then we can have the following analogous result:

4.3 Theorem

Let $\langle R_n(F), \circ \rangle$ be arhotrix regular subsemigroupof a matrix semigroup $\langle M_n(F), \cdot \rangle$ then it follows that

(i) L ($R_n(F)$)=L ($M_n(F)$) $\cap (R_n(F) \times R_n(F))$,

(ii) R ($R_n(F)$)=R ($M_n(F)$) $\cap (R_n(F) \times R_n(F))$,

(iii) H ($R_n(F)$)=H ($M_n(F)$) $\cap (R_n(F) \times R_n(F))$,

Furthermore, by Howie [12][Ex.2.6, no.19], we can havethe following result characterizing
L, R,andH in the semigroup $\langle R_n(F), \circ \rangle$.

4.4 Theorem (Characterization of L, R,and H relationsin the semigroup $\langle R_n(F), \circ \rangle$.

Let $A, B \in R_n(F)$, then

i. A L Bif and only if $im(A) = im(B)$
ii. ALBif and only if $\ker(A) = \ker(B)$
iii. H=L $\cap$ R

Proof
The proof follows directly from theorem 4.2.

To characterize theDrelation in $R_n(F)$, we observe that D $(R_n(F)) \subseteq$ D $(M_n(F))$. Consider

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

in $R_n(F)$. Then $rank(A) = rank(B) = 3 \Rightarrow (A, B) \in$ D $(M_n(F))$. Since, $rank(a_{ij}) = 1 \neq 2 = rank(b_{ij})$ and $rank(c_{kl}) = 2 \neq 1 = rank(d_{kl})$, by lemma 4.1,we can rightly say that $(A, B) \notin$ D $(R_n(F))$. Therefore, D $(R_n(F)) \subset$ D $(M_n(F))$ properly.
The next theorem follows from lemma 4.1.

4.5 Theorem (Characterization of Drelation in the semigroup $\langle R_n(F), \circ \rangle$.

ADB if and only if $rank(a_{ij}) = rank(b_{ij})$ *and* $rank(c_{kl}) = rank(d_{kl})$.

Proof

The prove follows from lemma 4.1 and Howie [12, Ex.2.6 (19)].

4.6 Corollary

$R_n(F)$ has exactly $\dfrac{(n+1)(n+3)}{4}$ D-classes

$D(0,0), D(0,1),..., D(0,t-1), D(1,0), D(1,1),..., D(t,0),..., D(t,t-1)$, where $t = \dfrac{n+1}{2}$ and $D(r,s) = \{A = \langle a_{ij}, c_{kl} \rangle \in R_n(F) \mid rank(a_{ij}) = r, rank(c_{kl}) = s\}$, with $(0 \le r \le t, 0 \le s \le t-1)$.

Let F be a finite field and set $h = |F|$. In the next theorem, we calculate the cardinality of all D-classesand we also calculate the number of L-classesand R-classescontained in $D(r,s)$ (that is the width and the height of the egg-box $D(r,s)$) and the cardinality of all H-classes within $D(r,s)$

For each r and s, $0 \le r \le t$ and $0 \le s \le t-1$ respectively, we let

$$V_n^{r+s} = \frac{(h^n - 1)(h^n - h)...(h^n - h^{r+s-1})}{(h^{r+s} - 1)(h^{r+s} - h)...(h^{r+s} - h^{r+s-1})}$$

denote the number of subspaces of dimension $r+s$ in F^n. Set $V_n^0 = 1$.

4.7 Theorem

Let r and s be arbitrary integers satisfying $0 \le r \le t$ and $0 \le s \le t-1$ respectively, and let F be a finite field, $h = |F|$.

1. The D-classes $D(r,s)$ contains exactly $V_n^{n-(r+s)}$ R-classes and V_n^{r+s} L-classes.
2. The cardinality of everyH-classes within $D(r,s)$ equal $(h^{r+s} - 1)(h^{r+s} - h)...(h^{r+s} - h^{r+s-1})$.
3. $|D(r,s)| = V_n^{r+s} V_n^{n-r+s} (h^{r+s} - 1)(h^{r+s} - h)...(h^{r+s} - h^{r+s-1})$

$= V_n^{n-r+s} (h^n - 1)(h^n - h)...(h^n - h^{r+s-1})$

Proof

Let $A, B \in D(r,s)$. Then, $rank(A) = rank(B) = r+s$ and$A$L$B \Leftrightarrow im(A) = im(B)$. And so, the number of distinct L-classes within $D(r,s)$ correspond to the number of distinct $im(A)$ for which $rank(A) = r+s$. This equal the number of distinct subspaces of F^n with dimension $r+s$.The counting of R-classes within $D(r,s)$ is done similarly. This proved the first statement.

To prove the second statement, we use the description of H-classeswhich gives that an H-classes within $D(r,s)$ is determined by specifying two subspaces, V_1 and V_2, of dimension $r+s$ and $n-(r+s)$ respectively. Indeed, H consist of all linear maps A such that $\mathrm{Im}(A)=V_1$ and $Ker(A)=V_2$. This means that the cardinality of every such H-classes equals the number of isomorphism between two subspaces from F^n of dimension $r+s$.

5.0 Conclusion

We have presented a construction of certain semigroup using a set of all rhotrices of the same size, with entries from an arbitrary field *F*,as an underlying set, together with the binary operation for rhotrix multiplication defined by Sani. We have identified the basic properties of this rhotrix semigroup and characterized its Green's relations. Furthermore, as an analogous to regular semigroup of square matrices, we have shown that the rhotrix semigroup $\langle R_n(F),\circ\rangle$ is also a regular semigroup and it is embedded in matrix semigroup $\langle M_n(F),\circ\rangle$. In the future work, it may be interesting to consider a number of topics for rhotrix semigroups. Such research topics include: finiteness condition for rhotrix semigroups, computational problems in rhotrix semigroups and motarlity in rhotrix semigroups *etc*.These are problem areas that need consideration.

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