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Cores of Euclidean Targets for a Class of Double-Delay Autonomous Control Systems

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Abstract

This paper established the convexity, compactness and subspace properties of cores of Euclidean targets for a class of double-delay autonomous linear control systems. The paper also revealed that the problem of controlling any initial endowment to a prescribed target and holding it there reduces to that of controlling the endowment to the core of the target with no further discussion about the problem as soon as this core is attained, since the right kind of behavior has been enforced on the initial endowment. The proof of the boundedness relied on the notion of asymptotic directions and other convex set properties, while that of the closedness appropriated a weak compactness argument.

Keywords: Asymptotic, Closedness, Convexity, Cores, Targets.

1.0 Introduction

The relationship between cores of targets and Euclidean controllability was introduced by Hajek [1], who examined the system $\dot{x} = Ax - p,\ p(t) \in P,\ x(\text{end}) \in T,$ where A is an $n \times n$ coefficient matrix, $P,$ the constraint set a compact convex non-void subset of the real n-dimensional Euclidean space $\mathbf{R}^n$, the target set, T, a closed convex non-void subset of $\mathbf{R}^n$, and $p : I \to P,$ admissible controls on I, a subset of $\mathbf{R}^+ = [0, \infty).$

By exploiting the analyticity and non-singularity of the fundamental matrices of the associated homogeneous system $\dot{x} = Ax,$ and using the notion of asymptotic directions and other convex set properties, he established that $\text{core}(T)$ is bounded if and only if $\text{rank}\left[M^T, A^T M^T, \cdots, \left(A^T\right)^{n-1} M^T \right] = n,$ for some $m \times n$ constant matrix, M. He indicated that the closeness of $\text{core}(T)$ could be achieved by using a weak compactness argument.

Ukwu [2] extended Hajek's results to a delay control system, with the major contribution being the varying of the technique for the boundedness of $\text{core}(T)$ due to the singularity of the solution matrices and certain other properties of such matrices. See also Ref. [2] and [3] for other results on cores. This paper gives further results on cores. In the sequel, the following questions are at the heart of the matter: under what conditions can a set of initial functions be driven to some prescribed targets in $\mathbf{R}^n$, and maintained there, thereafter by the implementation of some control procedure?, what are the initial endowments (core of target) that can be so steered?, is the set of these initial points compact?. These questions are well-posed and are applicable in the control of global economic growth, when the main consideration is the issue of possibilities for the control of the growth of capital stock from initial endowments to the desired ranges of values of the capital stock. It is clear that no firm has unlimited capacity to invest in growth. Thus the initial assets or capital stock that can be built up to prescribed levels of growth cannot be too big, and may be compact, indeed.

2.0 Theoretical Analysis

2.1 Definition of Systems of Interest with the Standing Hypotheses

We consider the double-delay autonomous control system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) + B\,u(t);\ t \ge 0 \qquad (1)$$

$$x(t) = \phi(t),\ t \in [-2h, 0],\ h > 0 \qquad (2)$$

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with initial function $\phi \in C\left(\left[-2h,0\right],\mathbf{R}^n\right)$ defined by:

$$\phi\left(s\right)=g\left(s\right),\ s\in[-2h,0);\ \phi\left(0\right)=g^0\in\mathbf{R}^n \qquad (3)$$

where $A_0, A_1, A_{2\,1}$ are $n\times n$ constant matrices and B is an $n\times m$ constant matrix $x\left(\tau\right)\in\mathbf{R}^n$ for $\tau\geq -2h$. The space of admissible controls is

$L_\infty^{loc}\left([0,\ \infty),\mathbf{R}^m\right): u(t)\in U$ a.e.,$t\in[0,\ \infty); U\subset\mathbf{R}^m$, a compact , convex set with $0\in$ interior of U

Denote this by $\Omega = L_\infty^{loc}\left([0,\infty),\,U\right)$. The target set H is a closed, convex, nonvoid subset of $\mathbf{R}^n$.

See [4] for an exposition on $\Omega = L_\infty^{loc}\left([0,\infty),\,U\right)$.

It is appropriate at this stage to state some relevant definitions, as well as discuss some preliminary notions from convex set theory as they relate to cores of Euclidean targets and then collect the results needed for the proofs of the main results.

2.2 Definition of Cores of Targets

The core of the target set $H\subset\mathbf{R}^n$ denoted by core (*H*) consists of all initial points $\phi\left(0\right)=g^0\in\mathbf{R}^n$, where $\phi\in C\left(\left[-2h,0\right],\,\mathbf{R}^n\right)$ for which there exists an admissible control *u* such that the solution (response) $x\left(t,\phi,u\right)$ of system (1) with $x_0=\phi$ as the initial function, satisfies $x\left(t,\phi,u\right)\in H,$ for all $t\geq 0$.

2.3 Definition of Asymptotic Directions of Convex Sets

Let K be a closed, convex set in $\mathbf{R}^n$. A vector $a\in\mathbf{R}^n$ is an asymptotic direction of K if for each $x\in K$ and all $\lambda\geq 0$, we have $x+\lambda a\in K$; that is, the half-ray issuing from x in the direction a lies entirely within K.

2.4 Definition of Sets of Asymptotic Directions of Convex Sets

The set $O^+\left(K\right)$ defined by:

$$O^+(K)=\left\{a\in\mathbf{R}^n : x+\lambda\, a\in K \text{ for every } \lambda\geq 0 \text{ and every } x\in K\right\} \qquad (4)$$

denotes the set of asymptotic directions of K. By definition 2.2, it consists of all asymptotic directions of K.

2.5 Lemma on the Convexity of the Set of Asymptotic Directions of a convex set

$O^+\left(K\right)$ is a convex cone containing the origin.

Proof

Let $a\in O^+\left(K\right)$. Then $x+\lambda a\in K$ for every $\lambda\geq 0$ and every $x\in K.$ Let $\mu\geq 0$. Then $x+\left(\lambda\mu\right)a\in K$, since $\lambda\mu\ \geq 0$. Therefore $x+\lambda\left(\mu a\right)\in K,$ for every $\lambda\geq 0$ and every $x\in K$,

showing that $\mu a\in O^+\left(K\right)$ if $a\in O^+\left(K\right)$. Thus, $O^+\left(K\right)$ is closed under nonnegative scalar multiplication. Therefore $O^+\left(K\right)$ is a cone.

Convexity: Let $a_1\ a_2\in O^{+,}\ \left(K\right)$ and $0\leq\lambda\leq 1$. Then

$$\left(1-\lambda\right)a_1+\lambda\, a_2+K=\left(1-\lambda\right)\left(a_1+K\right)+\lambda\left(a_2+K\right)\subset\left(1-\lambda\right)K+\lambda\, K=K$$

since $a_i+K\subset K,\ i=1,2$ by the definition of $O^+\left(K\right)$. Hence $\left(1-\lambda\right)a_1+\lambda\, a_2\in O^+\left(K\right)$. This proves that $O^+\left(K\right)$ is convex.

2.6 Lemma on Boundedness of Closed Convex sets, ([1], p.203)

A nonempty closed convex set K in $\mathbf{R}^n$ is bounded if and only ifzero is its only asymptotic direction; that is $O^+\left(K\right)=\left\{0\right\}$.

2.7 Lemma on Nonvoidsets expressed as Direct Sums

If a nonvoid set D is of the form $D=L+\hat{E}$, where $\hat{E}$ is bounded and L is a linear subspace of *D*, then L is the largest linear subspace of *D* and necessarily coincides with the set of asymptotic directions of *D*. Cf. ([1], p.204).

Further discussions on convex sets may be found in [5].

3.0 Results and Discussions

3.1 Existence, Uniqueness and Representations of solutions of related systems

If $\phi \in C([-h,0], E^n)$ and u is an admissible control in Ω, then there exists a unique solution of $\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t),$ for $t \geq 0,$ satisfying $x(t) = \phi(t)$, for $t \in [-h,0]$.

This solution is given by:

$$x(t, \phi, u) = x(t, \phi, 0) + \int_0^t Y(t-s) Bu(s) ds, \qquad (5)$$

where $Y(t)$ is a solution matrix solution of:

$$x'(t) = A_0 x(t) + A_1 x(t-h), t > 0 \qquad (6)$$

with:

$$Y(t) = \begin{cases} I_n; \ t = 0 \\ 0, \quad t < 0, \end{cases} \qquad (7)$$

and:

$$x(t, \phi, 0) = Y(t)\phi(0) + \int_{-h}^{0} Y(t-s-h) A_1 \phi(s) ds \qquad (8)$$

In general, the fundamental matrix $Y(t)$ may be singular for $t \geq 0$; it is of class C^∞ on $(jh, (j+1)h), j = 0, 1, ...$ and of bounded variation on compact intervals.

By the transformation:

$$x(t, \phi, 0) = \mathrm{T}(t)\phi(0) \qquad (9)$$

(9) becomes:

$$x(t, \phi, u) = \mathrm{T}(t)\, \phi(0) + \int_0^t Y(t-s)Bu(s)ds \qquad (10)$$

where $\mathrm{T}(t)$ has the following properties: $\mathrm{T}(t)$ is an operator defined on $C([-h,0], E^n)$, for $t \geq 0$.

i. The family $\{\mathrm{T}(t) : t \geq 0\}$ is a semi-group of linear transformations

ii. $\mathrm{T}(t)$ is bounded for each $t \geq 0$

iii. $\mathrm{T}(0) = I$ and $\mathrm{T}(t)$ is strongly continuous

iv. $\mathrm{T}(t)$ is completely continuous for $t \geq h$

See lemma 19.1 in [6].

Suppose $x(t, ; 0) : C \to E^n$ is a continuous linear operator, ([7], p. 345). The map $x(t, ; \ .) : C \times \Omega \to E^n$ in (10) can be written as $x(t, \phi, u) = x(t, \phi, o) + x(t, 0, u)$, for $(\phi, u) \in C \times \Omega$. This is immediate from (9), (10) and the properties of the family $\{\mathrm{T}(t) : t \geq 0\}$.

From the above theorem the following theorem is immediate.

3.2 Existence, Uniqueness and Representations of Solutions of System (1)

If $\phi \in C([-2h,0], \mathbf{R}^n)$ and u is an admissible control in Ω, then there exists a unique solution of $\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t),$ for $t \geq 0,$ satisfying $x(t) = \phi(t)$, for $t \in [-2h, 0]$.

This solution is given by: $x(t, \phi, u) = x(t, \phi, 0) + \int_0^t Y(t-s) Bu(s) ds,$ (11)

where $Y(t)$ is the fundamental matrix solution of:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h), t > 0 \qquad (12)$$

with

$$Y(t)=\begin{cases} I_n; \ t=0 \\ 0, \quad t<0, \end{cases} \tag{13}$$

and:

$$x(t,\phi,0)=Y(t)\phi(0)+\int_{-h}^{0} Y(t-s-h)A_1\phi(s)ds+\int_{-2h}^{0} Y(t-s-2h)A_1\phi(s)ds \tag{14}$$

Now, we are in a position to establish a sequence of results on core (H).

3.3 Lemma on nonvoidness of core (*H*)

If $0\in H$ and $0\in U$, then $0\in$ core (H). Hence core (H) is nonvoid.

Proof

Choose $\phi=0$ in $C([-2h,0],\mathbf{R}^n)$. Then $\phi(0)=0$ and $\phi(s)=0,\ s\in[-2h,0]$. If $0\in U$, then $u=0\in\Omega$ is an admissible control. From (14) we get $x(t,0,0)=0$. If $0\in H$, we conclude that $0\in$ core (H) and so core (H) is nonvoid.

The following theorem establishes the convexity and closedness of core (H).

3.4 Theorem on Convexity and Closedness of core (*H*)

Under the hypothesis on the control system (1), core (H) is convex and closed.

Proof

The proof will be realized from the convexity of Ω and H and an application of a Weak Compactness argument.

Convexity

Let $g_i^0\in$ core (H), i = 1, 2; Then $\phi_i(0)=g_i^0$ for some $\phi_i\in C([-2h,0],\mathbf{R}^n),\ i=1,2.$ Corresponding to ϕ_i there exist two admissible controls u_1,u_2 and two trajectories $x(t,\phi_1,u_1),\ x(t,\phi_2,u_2)$, such that $x(t,\phi_i,u_i)\in H$ for all $t\geq 0;\ i=1,2$. Let $0\leq\lambda\leq 1$. Then $\lambda x(t,\phi_1,u_1)+(1-\lambda)x(t,\phi_2,u_2)\in H$ for all $t\geq 0$, since H is convex.

But:

$$\begin{aligned}
&\lambda x(t,\phi_1,u_1)+(1-\lambda)x(t,\phi_2,u_2)\\
&=\lambda Y(t)\phi_1(0)+\lambda\int_{-h}^{0} Y(t-\tau-h)A_1\phi_1(\tau)d\tau+\lambda\int_{-2h}^{0} Y(t-\tau-2h)A_2\phi_1(\tau)d\tau\\
&+\lambda\int_0^t Y(t-\tau)Bu_1\\
&+(1-\lambda)Y(t)\phi_2(0)+(1-\lambda)\int_{-h}^{0} Y(t-\tau-h)A_1\phi_2(\tau)d\tau+(1-\lambda)\int_{-h}^{0} Y(t-\tau-2h)A_2\phi_2(\tau)d\tau\\
&+(1-\lambda)\int_0^t Y(t-\tau)Bu_2(\tau)d\tau
\end{aligned} \tag{15}$$

$$\begin{aligned}
&=Y(t)\left[\lambda\phi_1+(1-\lambda)\phi_2\right](0)\\
&=+\int_{-h}^{0} Y(t-\tau-h)A_1(\lambda\phi_1+(1-\lambda)\phi_2)(\tau)d\tau+\int_{-2h}^{0} Y(t-\tau-2h)A_2(\lambda\phi_1+(1-\lambda)\phi_2)(\tau)d\tau\\
&\quad+\lambda\int_0^t Y(t-\tau)B(\lambda u_1+(1-\lambda)u_2)(\tau)d\tau\in H,\ \text{for all }\ t\geq 0
\end{aligned} \tag{16}$$

Certainly, $\lambda\phi_1+(1-\lambda)\phi_2\in C([-2h,0],\mathbf{R}^n)$, by the convexity of $C([-2h,0],\mathbf{R}^n)$.

Also,
$\lambda\, u_1 + (1-\lambda)\, u_2 \in \Omega$, by the convexity of L_∞ and U . Hence
$\left(\lambda\, \phi_1 + (1-\lambda)\, \phi_2\right)(0) \in$ core (H); that is $\lambda\, g_1^0 + (1-\lambda) g_2^0 \in$ core (H) for any
$g_1{}^0,\, g_2{}^0 \in$ core (H) and $0 \le \lambda \le 1$. So, core (H) is convex.

<u>Closedness</u>

Consider a sequence of points $\{g_k^0\}_1^\infty$ in core (H) such that $\lim_{k\to\infty} g_k^0 = g^0$. Then, by the definition of core (H), there exist $\phi_k \in C\left([-2h,0], \mathbf{R}^n\right), k = 1,2,..$ for which $\phi_k(0) = g_k^0$. Let $\{u_k\}_1^\infty$ be an appropriate sequence of admissible controls corresponding to $\{\phi_k\}_1^\infty$ for which $x(t, \phi_k, u_k) \in H$ for all $t \ge 0$. Now the class of admissible controls Ω is just the closed balls in $L_\infty^{loc}\left([0, \infty]\right)$ of some finite radius r ; hence by Banach-Alaoglu theorem [8, 9], Ω is weak-star compact (denoted w*-compact) and convex. Consequently, there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ of $\{\mathrm{u_k}\}_1^\infty$ such that $u_{k_j} \xrightarrow{w^*} u$ for some $u \in \Omega$.

Thus

$$\lim_{j\to\infty}\int_0^t Y(t-\tau)Bu_{k_j}(\tau)d\tau = \int_0^t Y(t-\tau)Bu(\tau)d\tau, \text{ for } 0 \le t < \infty.$$

Let $\{\phi_{k_j}\}_{j=1}^\infty$ be a subsequence of $\{\phi_k\}_1^\infty$ corresponding to $\{u_{k_j}\}_{j=1}^\infty$ for which

$\phi_{k_j}(0) = g_{k_j}^0 \in \mathrm{Core}(H)$ Then

$$x(t,\phi_{k_j},u_{k_j}) = T(t)\phi_{k_j}(0) + \int_0^t Y(t-\tau)Bu_{k_j}(\tau)d\tau \in H, \text{for all } t \ge 0.$$

By virtue of the closednessof H , we have $\lim_{j\to\infty} x\left(t, \phi_{k_j}, u_{k_j}\right) \in H,\ t \ge 0$.

Therefore, $\lim_{j\to\infty} T(t)\phi_{k_j}(0) + \lim_{j\to\infty}\int_0^t Y(t-\tau)Bu_{k_j}(\tau)d\tau \in H, \text{for all } t \ge 0.$

But $\lim_{j\to\infty}\int_0^t Y(t-\tau)Bu_{k_j}(\tau)d\tau = \int_0^t Y(t-\tau)Bu(\tau)d\tau$

$$T(t)\lim_{j\to\infty} g_{k_j}^0 + \int_0^t Y(t-\tau)Bu(\tau)d\tau \in H, \text{for all } t \ge 0, \text{for some } u \in \Omega.$$

Hence,

Therefore $\lim_{j\to\infty} g_{k_j}^0 \in \mathrm{core}(H).$ But:

$$\lim_{j\to\infty} g_{k_j}^0 = \lim_{k\to\infty} g_k^0 = g^0 \tag{17}$$

This proves that the limit of any convergent sequence of points in core (H) is also inCore (H); hencecore (H) is closed.
The next result relates the asymptotic directions of H to those of core (H) .

3.5 Lemma relating the Asymptotic Directions of *H* to those of Core(*H*)

A vector $a \in \mathbf{R}^n$ is an asymptotic direction of core (H) if and only if $Y(t)a$ is an asymptotic direction of H .

Proof

Let $Y(t)a$ be an asymptotic direction of H for some vector $a \in \mathbf{R}^n$. Then:

$$H + \theta Y(t)a \subset H \tag{18}$$

for each $\theta \geq 0$. Take $g^0 \in$ core (H) and a corresponding $\phi \in C([-2h, 0], \mathbf{R}^n)$ such that $\phi(0) = g^0$. Let $u \in \Omega$ be an admissible control, which holds the response $x(t, \phi, u)$ within H. Then:

$$x(t,\phi,u) = Y(t)\phi(0) + \int_{-h}^{0} Y(t-\tau-h)A_1\phi(\tau)\,d\tau + \int_{-2h}^{0} Y(t-\tau-2h)A_2\phi(\tau)\,d\tau$$

$$+\int_0^t Y(t-\tau)Bu(\tau)\,d\tau \in H, \forall t \geq 0 \tag{19}$$

$$x(t,\phi,u) = Y(t)\left[g^0 + \theta a\right] + \int_{-h}^{0} Y(t-\tau-h)A_1\phi(\tau)\,d\tau + \int_{-2h}^{0} Y(t-\tau-2h)A_2\phi(\tau)\,d\tau$$

$$+\int_0^t Y(t-\tau)Bu(\tau)\,d\tau$$

$$= Y(t)g^0 + \int_{-h}^{0} Y(t-\tau-h)A_1\phi(\tau)\,d\tau + \int_{-2h}^{0} Y(t-\tau-2h)A_2\phi(\tau)\,d\tau$$

$$+\int_0^t Y(t-\tau)Bu(\tau)\,d\tau + \theta Y(t)a \tag{20}$$

$$\in H + \theta Y(t)H \subset H, \tag{21}$$

by (18) and (19). We deduce immediately from (21) that $g^0 + \theta a \in$ core (H), for each $\theta \geq 0$. Hence, a is an asymptotic direction of core (H).

Conversely suppose $a \in \mathbf{R}^n$ is an asymptotic direction of core (H). Let $g^0 \in$ core (H). Then $g^0 + \theta a \in$ core (H) for each $\theta \geq 0$. Hence, there exists an admissible control u ndafunction $\psi \in C([-2h, 0], \mathbf{R}^n)$ such that the solution $x(t,\psi,u)$, with $x(\psi,u) = \psi, \psi(0) = g^0 + \theta a$ satisfies $x(t,\psi,u) \in H, \forall t \geq 0.$ Therefore we have

$$x(t,\psi,u) = Y(t)\left[g^0 + \theta a\right] + \int_{-h}^{0} Y(t-\tau-h)A_1\psi(\tau)\,d\tau + \int_{-2h}^{0} Y(t-\tau-2h)A_2\psi(\tau)\,d\tau$$

$$+\int_0^t Y(t-\tau)Bu(\tau)\,d\tau = b_\theta, \text{for some } b_\theta \in H, \text{for } t \geq 0.$$

Let $\theta > 0$; divide through by θ and take the limits of both sides of the expression for $x(t,\psi,u)$ as $\theta \to \infty$ to get

$$\lim_{\theta\to 0} \frac{1}{\theta} Y(t)g^0 + \lim_{\theta\to 0} Y(t)a$$

$$+\lim_{\theta\to 0} \frac{1}{\theta}\int_{-h}^{0} Y(t-\tau-h)A_1\psi(\tau)\,d\tau + \lim_{\theta\to 0} \frac{1}{\theta}\int_{-2h}^{0} Y(t-\tau-2h)A_2\psi(\tau)\,d\tau + \lim_{\theta\to 0} \frac{1}{\theta}\int_0^t Y(t-\tau)Bu(\tau)\,d\tau$$

$$= \lim_{\theta\to 0} \frac{1}{\theta} b_\theta.$$

Now $\lim_{\theta\to\infty} \frac{1}{\theta} Y(t)g^0 = 0,$ since $Y(t)g^0$ is independent of θ. Also the limit of above integrals is zero since the integrals are bounded for fixed $t,\ 0 \leq t < \infty$.

Therefore:

$$Y(t)a = \lim_{\theta\to\infty} \frac{1}{\theta} b_\theta \tag{22}$$

for some $b_\theta \in H$. Let $c \in H$ and let $\lambda \geq 0$. We must show that $c + \lambda Y(t)a \in H.$

If λ is fixed and $\theta \geq \lambda$, then $0 \leq \frac{\lambda}{\theta} \leq 1$ for $\theta > 0$. Consequently, the convexity of H implies that $\left(1 - \frac{\lambda}{a}\right)c + \frac{\lambda}{\theta} b_\theta \in H$. Now $\lim_{\theta \to \infty}\left[\left(1 - \frac{\lambda}{\theta}\right)c + \frac{\lambda}{\theta} b_\theta\right] \in H$, because H is closed. Therefore we have $c + \lambda \lim_{\theta \to \infty} \frac{\lambda}{\theta} b_\theta \in H$.This shows that $c + \lambda Y(t)a \in H$, by (22). It follows immediately from definition 2.2that $Y(t)a$ is an asymptotic direction of H, as required.

We proceed to establish another useful property of core (H).

3.6 Theorem on useful properties of Core(*H*)

Core (core (H)) = core (H)

<u>Proof</u>

Let $g^0 \in$ core (H). Then, $\phi(0) = g^0$ for some $\phi \in C([-2h, 0], \mathbf{R}^n)$. Thus there exists an admissible control $u \in \Omega$ such that $x(t, \phi, u) \in H$ for all $t \geq 0.$ Fix a time $\bar{t} > 0$; then $x_{\bar{t}}(\phi, u)$ serves as the initial function for a response starting at $\bar{t}$ with the same control u and with the initial point $x_{\bar{t}}(\phi, u)(0) = x(\bar{t}, \phi, u) \in H$.

Now $x(t, \bar{t}, x_{\bar{t}}(\phi, u), u) = x(t, \phi, u) \in H$ for all $t \geq \bar{t} \geq 0$, showing that $x_{\bar{t}}(\phi, u)(0) \in$ core (H); that is, $x(\bar{t}, \phi, u) \in$ core (H).

Applying the definition of core to the new target core (H), we deduce that $\phi(0) \in$ core (core (H)). But $\phi(0) = g^0$ is arbitrary in core (H). Therefore core (H) $\subset$ core (core (H)). The reverse inclusion $\text{core}(\text{core}(H)) \subseteq \text{core}(H)$ is immediate. Therefore, core (core $(\text{Core }(H))$) = core (H).

The implicationof above result is that the problem of controlling any initial endowment to a prescribed target and holding it there reduces to that of controlling the endowment to the core of the target with no further discussion about the problem as soon as this core is attained, since the right kind of behavior has been enforced on the initial endowment.

3.7 Theorem on Core (*H*) as a Subspace

Consider the control system (1) with its standing hypotheses. Let the target set H be of the form. $H = L + D$, where $L = \{x \in \mathbf{R}^n : Mx = 0\}$ for some $m \times n$ matrix $M,$ and some bounded, convex subset D of $\mathbf{R}^n$ with $0 \in D$. Assume that $0 \in U$ and $0 \in H$. Then:

$$O^+(\text{core}(H)) = \bigcap_{t \geq 0}\left\{\bar{x} \in O^+(H) : MY(t)\bar{x} = 0\right\},$$

and it is the largest subspace of $O^+(H)$ trapped in $O^+(H)$ under the map $t \to Y(t)$, for each $t \geq 0.$

<u>Proof</u>

By lemma 2.6, $O^+(H) = L.$ If $\bar{x} \in O^+(\text{core}(H)),$ then by lemma 3.4, $Y(t)\bar{x} \in O^+(H)$ for $t \geq 0.$ Hence $MY(t)\bar{x} = 0,$ for $t \geq 0.$ In particular, at $t = 0,$ we have $Y(t) = Y(0) = I_n,$ yielding $M\bar{x} = 0;$ this shows $\bar{x} \in O^+(H).$ The results: $MY(t)\bar{x} = 0,$ for $t \geq 0,$ and $\bar{x} \in O^+(H)$ imply that $\bar{x} \in \left\{y \in O^+(H) : MY(t)y = 0\right\}, \forall t \geq 0.$ Since $\bar{x}$ is arbitrary in $O^+(\text{core}(H)),$ we deduce immediately that

$$O^+(\text{core}(H)) \subset \bigcap_{t \geq 0}\left\{y \in O^+(H) : MY(t)y = 0\right\}.$$

For the proof of the reverse inclusion, let $\bar{x} \in \left\{y \in O^+(H) : MY(t)y = 0\right\}, \forall t \geq 0.$ Then

$MY(t)\bar{x} = 0, \forall t \geq 0.$ Hence $Y(t)\bar{x} \in O^{+}(H), \forall t \geq 0.$ The result $\bar{x} \in O^{+}(\text{core}(H))$ is immediate from lemma 3.4. We deduce from the arbitrariness of $\bar{x}$ in $\bigcap_{t\geq 0}\{y \in O^{+}(H) : MY(t)y = 0\}$, that $\bigcap_{t\geq 0}\{y \in O^{+}(H) : MY(t)y = 0\} \subset O^{+}(\text{core}(H))$

Hence $O^{+}(\text{core}(H)) = \bigcap_{t\geq 0}\{\bar{x} \in O^{+}(H) : MY(t)\bar{x} = 0\},$

That $O^{+}(\text{core}(H))$ is a subspace follows from the fact that if $\bar{x}, \bar{y} \in O^{+}(\text{core}(H))$ and $\alpha, \beta \in \mathbf{R}$, then $\alpha\bar{x} + \beta\bar{y} \in O^{+}(H) = L$, L being a linear space. So $MY(t)[\alpha\bar{x} + \beta\bar{y}] = \alpha MY(t) + \beta MY(t) = \alpha.0 + \beta.0 = 0,$ since $\bar{x}, \bar{y} \in L$. Therefore $\alpha\,\bar{x} + \beta\,\bar{y} \in O^{+}(\text{core}(H))$, showing that $O^{+}(\text{core}(H))$ is a subspace of $O^{+}(H)$.To show that $O^{+}(\text{core}(H))$ is trapped in $O^{+}(H)$ under $Y(t)$ for each $t \geq 0$, let $\bar{x} \in O^{+}(\text{core}(H)$. Then $Y(t)\bar{x} \in O^{+}(H)$, by lemma 3.4. So $\bar{x} \in \{y \in O^{+}(H) : MY(t)y = 0\}$ and

$O^{+}(\text{core}(H)) \subset \{V \subset O^{+}(H) : Y(t)V \subset O^{+}(H)\}$. Hence $O^{+}(\text{core}(H))$ is trapped in $O^{+}(H)$ under the map $Y(t)$ for each $t \geq 0$. $\alpha, \beta \in \mathbf{R} \Rightarrow \alpha\bar{x} + \beta\bar{y} \in O^{+}(H) = L.$

If W is another subspace of $O^{+}(H)$ trapped in $O^{+}(H)$ under the map $Y(t)$ for each $t \geq 0$, then $Y(t)w \in 0^{+}(H)$ for all $w \in W$. Hence $w \in O^{+}(\text{core}(H))$, by definition 2.3 and lemma3.4. The conclusion $W \subseteq O^{+}(\text{core}(H))$ is immediate. This completes the proof of the theorem.

4.0 Conclusion

This article investigated some subspace and topological properties of cores of targets of a class of double-delay autonomous linear control systems, effectively extending the relevant results in [1] and [2] with economic interpretations, thereby providing a useful tool for the investigation and interpretation of Euclidean controllability.

References

[1] Hajek, O. (1974). Cores *of targets in Linear Control systems. Math. Systems theory, Vol. 8, No. 3, pp. 203 – 206,* Springer-Verlag, New York.

[2] Ukwu, C. (1987). *Compactness of cores of targets for linear delay systems., J. Math. Analy. and Appl., Vol. 125, No. 2, August 1.* pp. 323-330.

[3] Iheagwam, V. A. (2003). *Characterization of Cores of Targets for Ordinary systems with Distributed Delays in Control. JNMS, 2003, Vol. 22,* Pp. 23-34.

[4] Chidume, C. (2007). *Applicable functional analysis*. The Abdus Salam, International Centre for Theoretical Physics, Trieste, Italy.

[5] Rockafeller, R. T. (1970). *Convex analysis. princeton*Univ. Press, Princeton, New Jersey.

[6] Hale, J. K. (1977). *Theory of functional differential equations. Applied Mathematical Science, Vol. 3, Springer-Verlag,* New York.

[7] Chukwu, E. N. (1992). *Stability and time-optimal control of hereditary systems*. Academic Press, New York.

[8] Hermes, M. and LaSalle, J. P. (1969). *Functional analysis and time-optimal control.* Academic Press, New York.

[9] Knowles, G.(1981). *Introduction to applied optimal control.* Academic Press, New York.