Journal of the Nigerian Association of Mathematical Physics
Volume27, (July, 2014), pp 49 – 60
© J. of NAMP

Interactions Amongst Determining Matrices, Partials of Indices of control Systems Matrices and Systems Coefficients for a Classof Double – Delay Control Systems

Ukwu Chukwunenye

Department of Mathematics, University of Jos, P.M.B 2084
Jos, Nigeria. E-mail: cukwu@hotmail.com

Abstract

This paper obtained various relationships among determining matrices, partial derivatives of indices of control systems matrices of all orders,as well as their relationships with systems coefficients for a class of double – delay autonomous linear differential systems through a sequence of lemmas, theorems, corollaries and the exploitation of key facts about permutations. The utility of these relationships is for the most part, in the investigation of Euclidean controllability.

The proofs were achieved using ingenious combinations of summation notations, the multinomial distribution, greatest integer functions, change of variables techniques and deft deployment of skills in the differentiation of certain matrix functions of several variables.

1.0 Introduction

The importance of the relationships among determining matrices, indices of control systems matricesand systems coefficient derives from the fact that these relationships pave the way for the determination of Euclidean controllability and compactness of cores of Euclidean targets. This paper brings fresh perspectives to bear on such relationships, as reflectedin theorems 2.3, 2.4 and corollaries 3.1 through 3.3 to say the least.

1.1 Identification of Work-based Double-delay Autonomous Control System

We consider the double-delay autonomous control system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) + B\,u(t);\ t \ge 0 \qquad (1.1)$$

$$x(t) = \phi(t),\ t \in [-2h, 0],\ h > 0 \qquad (1.2)$$

where A_0, A_1, A_2 are $n \times n$ constant matrices with real entries, B is an $n \times m$ constant matrix with real entries. The initial function ϕ is in $C([-2h, 0], \mathbf{R}^n)$, the space of continuous functions from $[-2h, 0]$ into the real *n*-dimension Euclidean space, $\mathbf{R}^n$ with norm defined by $\|\phi\| = \sup_{t \in [-2h, 0]} |\phi(t)|$, (the sup norm). The control *u* is in the space $L_\infty([0, t_1], \mathbf{R}^n)$, the space of essentially bounded measurable functions taking $[0, t_1]$ into $\mathbf{R}^n$ with norm $\|\phi\| = ess\sup_{t \in [0, t_1]} |u(t)|$.

Any control $u \in L_\infty([0, t_1], \mathbf{R}^n)$ will be referred to as an admissible control. For full discussion on the spaces C^{p-1} and L_p (or L^p), $p \in \{1, 2, ..., \infty\}$, see [1] and [2] and [3].

Corresponding author: Ukwu Chukwunenye E-mail: cukwu@hotmail.com, Tel.: +2347052765751

1.2 Preliminaries on the Partial Derivatives $\frac{\partial^k X(\tau, t)}{\partial \tau^k}, k = 0, 1, \cdots$

Let $t, \tau \in [0, t_1]$. For fixed t, let $\tau \to X(\tau, t)$ satisfy the matrix differential equation

$$\frac{\partial}{\partial \tau} X(\tau, t) = -X(\tau, t) A_0 - X(\tau + h, t) A_1 - X(\tau + 2h, t) A_2 \qquad (1.3)$$

for $0 < \tau < t, \tau \neq t - kh, k = 0, 1, \ldots$ where $X(\tau, t) = \begin{cases} I_n; & \tau = t \\ 0; & \tau > t \end{cases}$

See [4], [5] and [6] for properties of $X(t, \tau)$. Of particular importance is the fact that $\tau \to X(\tau, t)$ is analytic on the intervals $(t_1 - (j+1)h, t_1 - jh), j = 0, 1, \ldots, t_1 - (j+1)h > 0$. Any such $\tau \in (t_1 - (j+1)h, t_1 - jh)$ is called a regular point of $\tau \to X(t, \tau)$. Let $X^{(k)}(\tau, t)$ denote $\frac{\partial^k}{\partial \tau^k} X(\tau, t_1)$, the k^{th} partial derivative of $X(\tau, t_1)$ with respect to τ, where τ is in $(t_1 - (j+1)h, t_1 - jh); j = 0, 1, \ldots, r$, for some integer r such that $t_1 - (r+1)h > 0$. Write $X^{(k+1)}(\tau, t_1) = \frac{\delta}{\delta \tau} X^k(\tau, t_1)$.

Define

$$\Delta X^{(k)}(t_1 - jh, t_1) = X^{(k)}\left(t_1, (t_1 - jh)^-, t_1\right) - X^{(k)}\left((t_1 - jh)^+, t_1\right), \qquad (1.4)$$

for $k = 0, 1, \ldots; j = 0, 1, \ldots; t_1 - jh > 0,$

where $X^{(k)}\left((t_1 - jh)^-, t_1\right)$ and $X^{(k)}\left(t_1, (t_1 - jh)^+, t_1\right)$ denote respectively the left and right hand limits of $X^{(k)}(\tau, t_1)$ at $\tau = t_1 - jh$. Hence

$$X^{(k)}\left((t_1 - jh)^-, t_1\right) = \lim_{\substack{\tau \to t_1 - jh \\ t_1 - (j+1)h < \tau < t_1 - jh}} X^{(k)}(\tau, t_1) \qquad (1.5)$$

$$X^{(k)}\left((t_1 - jh)^+, t_1\right) = \lim_{\substack{\tau \to t_1 - jh \\ t_1 - jh < \tau < t_1 - (j-1)h}} X^{(k)}(\tau, t_1) \qquad (1.6)$$

1.3 Definition, Existence and Uniqueness of Determining Matrices for System (1.1)

Let Q_k (s) be then $n \times n$ matrix function defined by

$$Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s - h) + A_2 Q_{k-1}(s - 2h) \qquad (1.7)$$

for $k = 1, 2, \cdots; s > 0,$ with initial conditions:

$$Q_0(0) = I_n \qquad (1.8)$$

$$Q_0(s) = 0; s \neq 0 \qquad (1.9)$$

These initial conditions guarantee the unique solvability of (1.7). Cf. [7]

2.0 Theoretical Framework

2.1 Theorem relating $\Delta X(t_1 - jh, t_1)$ to $Q_k(jh)$

$$\Delta X^{(k)}\left(t_1 - jh, t_1\right) = (-1)^k Q_k(jh), \forall j : t_1 - jh > 0. \qquad (2.1)$$

Proof

If $k = 0$, then $\Delta X^{(k)}(t_1 - jh, t_1) = \Delta X(t_1 - jh, t_1) = I_n \operatorname{sgn}(\max\{0, 1 - j\})$

$$= Q_0(jh) = (-1)^k Q_k(jh) = \begin{cases} I_n, \text{ if } j = 0 \\ 0, \text{ otherwise} \end{cases}$$

If $k = 1$, then we have

$$\Delta X^{(1)}(t_1 - jh, t_1) = X^{(1)}((t_1 - jh)^-, t_1) - X^{(1)}((t_1 - jh)^+, t_1)$$

$$= -X((t_1 - jh)^-, t_1)A_0 - X((t_1 - [j-1]h)^-, t_1)A_1 - X((t_1 - (j-2)h)^-, t_1)A_2$$

$$-[-X((t_1 - jh)^+, t_1)A_0 - X((t_1 - [j-1]h)^+, t_1)A_1 - X((t_1 - [j-2]h)^+, t_1)A_2]$$

$$= -[\Delta X(t_1 - jh, t_1)A_0 + \Delta X((t_1 - [j-1]h, t_1)A_1 + \Delta X((t_1 - [j-2]h), t_1)]A_2$$

$$= \begin{cases} -A_0 = (-1)^1 A_0, \text{ if j } = 0 \\ -A_1 = (-1)^1 A_{1,} \text{ if j } = 1 \\ -A_2 = (-1)^1 A_2, \text{ if j } = 2 \\ 0 \quad \text{ if j } \geq 3 \end{cases}$$

$$= (-1)^1 Q_1(jh) = (-1)^1 A_j \operatorname{sgn}(\max\{0, 3 - j\}) \qquad (2.2)$$

So the theorem is valid for $k \in \{0, 1\}$.
The rest of the proof by induction on k. Assume that the theorem is valid for $2 \leq k \leq n$, for some integer n. Then

$$\Delta X^{(n+1)}(t_1 - jh, t_1) = X^{(n+1)}((t_1 - jh)^-, t_1) - X^{(n+1)}((t_1 - jh)^+, t_1)$$

$$= \left[\frac{\partial}{\partial \tau} X^{(n)}(\tau, t_1)\right]\Big|_{\tau=(t_1-jh)^-} - \left[\frac{\partial}{\partial \tau} X^{(n)}(\tau, t_1)\right]\Big|_{\tau=(t_1-jh)^+}$$

$$= X^{(n)}((t_1 - jh)^-, t_1)A_0 - X^{(n)}((t_1 - [j-1]h)^-, t_1)A_1$$

$$-X((t_1 - [j-2]h)^-, t_1)A_2$$

$$+ X^{(n)}((t_1 - jh)^+, t_1)A_0 + X^{(n)}((t_1 - [j-1]h)^+, t_1)A_1$$

$$+ X^{(n)}((t_1 - [j-2]h)^+, t_1)A_2$$

$$= -\begin{bmatrix} \Delta X^{(n)}(t_1 - jh, t_1)A_0 + \Delta X^{(n)}(t_1 - [j-1]h, t_1)A_1 \\ +\Delta X^{(n)}((t_1 - [j-2]h, t_1)A_2 \end{bmatrix}$$

$$= (-1)^{n+1}[Q_n(jh)A_0 + Q_n([j-1]h)A_1 + Q_n([j-2]h)A_2]$$

(by the induction hypothesis)

$= (-1)^{n+1} Q_{n+1}(jh)$, (by the proof of theorem. 3.1 or 3.2 of [8]with 'leading'replaced by 'trailing').

Thus, the theorem is valid for $k = n + 1$ and hence valid for every non-negative integer k and for all j: $t_1 - j$h > 0.

$$Let\ \psi(c, \tau) = c^T X\left(\tau, t_1\right)B, c \in \mathbf{R}^n$$

andlet $\Delta \psi^{(k)}(c,\tau) = \psi^{(k)}(c,\tau^-) - \psi^{(k)}(c,\tau^+)$, for $\tau \in (0, \infty)$,
where $\psi^{(k)}(c,\tau) = \frac{\partial^k}{\partial\tau^k}\psi(c,\tau)$ and $(.)^T$ denotes the transpose of $(.)$.

2.2 Corollary to Theorem 2.1

$\Delta \psi^{(k)}(c, t_1 - jh) = (-1)^k c^T Q_k(jh)B$, for $k = 0, 1, \ldots$; and $j : t_1 - jh > 0$ (2.3)

Proof

Let j be a non-negative integer such that $t_1 - jh > 0$. Then

$\Delta \psi^{(k)}(c, t_1 - jh) = c^T \Delta X^{(k)}\left(t_1 - jh, t_1\right)B = c^T(-1)^k Q_k(jh)B$, (by theorem 2.1)

$= (-1)^k c^T Q_k(jh)B$, as desired.

2.3 Theorem relating $\left(\sum_{i=0}^{2}\mu_i A_i\right)^k$ to $Q_k(jh)$ involving certain evaluations at $\mu = (\mu_0, \mu_1, \mu_2)^T = 0$

For any real variables μ_0, μ_1, μ_2 and for any integer $k \geq 0$,

$$\left(\sum_{i=0}^{2}\mu_i A_i\right)^k = \sum_{j=0}^{2k}\sum_{r=0}^{\left(\left[\frac{2k-j}{2}\right]\right)}\mu_0^r \mu_1^{2k-j-2r}\mu_2^{r+j-k}Q_k(jh)\Big|\mu = 0 \quad (2.4)$$

in all permutation terms in $\mu_0^r, \mu_1^{2k-j-2r}\mu_0^{r+j-k}$, involving $A_0^{r_0}A_1^{r_1}A_2^{r_2}$, for which $(r_0, r_1, r_2) \neq (r, 2k-j-2r, r+j-k)$, where $\mu = (\mu_0, \mu_1, \mu_2)$ and all superscripts are nonnegative.

Proof

$k = 0 \Rightarrow j = 0 \Rightarrow r = 0 \Rightarrow$ rhs $= I_n =$ lhs; $k = 1 \Rightarrow j \in \{0, 1, 2\}$
$j = 0, r = 0 \Rightarrow$ rhs is infeasible and hence may be set equal to 0
$j = 0, r = 1 \Rightarrow$ rhs $= \mu_0 Q_1(0) = \mu_0 A_0$; $j = 1 \Rightarrow r = 0 \Rightarrow$ rhs $= \mu_1 Q_1(h) = \mu_1 A_1$
$j = 2 \Rightarrow r = 0 \Rightarrow$ rhs $= \mu_2 Q_1(2h) = \mu_2 A_2$
Adding up all the feasible contingencies we obtain

rhs$=\mu_0 A_o + \mu_1 A_1 + \mu_2 A_2 = \sum_{i=0}^{2}\mu_i A_i$. So, the lemma is valid for $k \in \{0, 1\}$.

Let us examine the case $k = 2$: $k = 2 \Rightarrow j \in \{0, 1, 2, 3, 4\}$.
$j = 0 \Rightarrow r \in \{0, 1, 2\}$; $j = 0, r \in \{0, 1,\} \Rightarrow$ rhs is infeasible.
$j = 0, r = 2 \Rightarrow$ rhs $= \mu_0^2 Q_2(0) = \mu_0^2 A_0^2$; $j = 1 \Rightarrow r \in \{0, 1,\}$; $j = 1, r = 0 \Rightarrow$ rhs is infeasible.
$j = 1, r = 1 \Rightarrow$ rhs $= \mu_0 \mu_1 Q_2(h) = \mu_0 \mu_1 [A_0 A_1, + A_0 A_1]$
$j = 2 \Rightarrow r \in \{0, 1\}$; $j = 2, r = 0 \Rightarrow$ rhs $= \mu_1^2 Q_2(2h)$
$\Rightarrow$ rhs $= \mu_1^2 Q_2(2h) = \mu_1^2 [A_0 A_2 + A_2 A_0 + A_1^2]$
$j = 2, r = 1 \Rightarrow$ rhs $= \mu_0 \mu_2 Q_2(2h) = \mu_0 \mu_2 [A_0 A_2 + A_2 A_0 + A_1^2]$
$j = 3 \Rightarrow r = 0 \Rightarrow$ rhs $= \mu_1 \mu_2 Q_2(3h) = \mu_1 \mu_2 [A_1 A_2 + A_2 A_1]$
$j = 4 \Rightarrow r \Rightarrow$ rhs $= \mu_1^2 Q_2(4h) = \mu_2^2 A_2^2$
Set $\mu_1 = 0$ in the term $\mu_1^2[A_0 A_2, + A_2 A_2]$; set $\mu_0 = \mu_2 = 0$ in the term $\mu_0 \mu_2 A_1^2$
Add up the feasible cases with the indicated evaluations to get
rhs$=\mu_0^2 A_0^2 + \mu_0 \mu_1 [A_0 A_1 + A_1 A_0] + \mu_1^2 A_1^2 + \mu_0 \mu_2 [A_0 A_2 + A_2 A_0]$

$+ \mu_1 \mu_2 [A_1 A_2, + A_2 A_1] + \mu_2^2 A_2^2 = \left[\sum_{i=0}^{2}\mu_i A_i\right]^2 =$ lhs

$k = 3 \Rightarrow j \in \{0, 1, \ldots, 6\}$, $r \in \{0, \ldots \left[\left[\frac{6-j}{2}\right]\right]$; $j = 0 \Rightarrow r \in \{0, 1, 2, 3\} \Rightarrow r = 0, 1, 2$ or 3

$j = 0, r \in \{0, 1, 2\} \Rightarrow$ rhs is infeasible, since $r + j - k < 0$.

$j = 0, r = 3 \Rightarrow$ rhs = = $\mu_0^3 Q_3(0) = \mu_0^3 A_0^3$, by lemma 2.5 of [8]. We are done with $j = 0$.
$j = 1 \Rightarrow r \epsilon \{0, 1\} \Rightarrow$ rhs is infeasible;
$j = 1, r = 2 \Rightarrow rhs = \mu_0^2 \mu_1 Q_3(h) = \mu_0^2 \mu_1 [A_0^2 A_1 + A_0 A_1 A_0 + A_1 A_0^2]$.
We are done with $j = 1$.
$j = 2 \Rightarrow r \epsilon \{0, 1, 2\}$; $j = 2, r = 0 \Rightarrow$ rhs is infeasible.
$j = 2, r = 1 \Rightarrow$ rhs = $\mu_0 \mu_1^2 Q_3(2h) = \mu_0 \mu_1^2 \begin{bmatrix} A_0^2 A_2 + A_2 A_0^2 + A_0 A_2 A_0 \\ + A_0 A_1^2 + A_1 A_0 A_1 + A_1^2 A_0 \end{bmatrix}$
$j = 2, r = 2 \Rightarrow$ rhs = $\mu_0^2 \mu_0 Q_3(2h) = \mu_0^2 \mu_2 \begin{bmatrix} A_0^2 A_2 + A_2 A_0^2 + A_0 A_2 A_0 \\ + A_0 A_1^2 + A_1 A_0 A_1 + A_1^2 A_0 \end{bmatrix}$.
We are done with $j = 2$.
$j = 3 \Rightarrow r \epsilon \{0, 1\}$; $j = 3, r = 0 \Rightarrow$ rhs = $\mu_1^3 Q_3(3h)$
$= \mu_1^3 \begin{bmatrix} A_0 A_1 A_2 + A_0 A_2 A_1 + A_1 A_0 A_2 \\ + A_1 A_2 A_0 + A_2 A_0 A_1 + A_2 A_1 A_0 + A_1^3 \end{bmatrix}$.
$j = 3, r = 1 \Rightarrow rhs = \mu_0 \mu_1 \mu_2 Q_3(3h)$
$= \mu_0 \mu_1 \mu_2 \begin{bmatrix} A_0 A_1 A_2 + A_0 A_2 A_1 + A_1 A_0 A_2 + A_1 A_2 A_0 \\ + A_2 A_0 A_1 + A_2 A_1 A_0 + A_1^3 \end{bmatrix}$
We are done with $j = 3$.
$j = 4 \Rightarrow r \epsilon \{0, 1\}$; $j = 4, r = 0 \Rightarrow$ rhs = $\mu_1^2 \mu_2 Q_3(4h)$
$= \mu_1^2 \mu_2 \begin{bmatrix} A_0 A_2^2 + A_2 A_0 A_2 + A_2^2 A_0 \\ + A_1^2 A_2 + A_1 A_2 A_1 + A_2 A_1^2 \end{bmatrix}$
$j = 4, r = 1 \Rightarrow$ rhs = $\mu_0 \mu_2^2 Q_3(4h)$
$= \mu_0 \mu_2^2 [A_0 A_2^2 + A_2 A_0 A_2 + A_2^2 A_0 + A_1^2 A_2 + A_1 A_2 A_1 + A_2 A_1^2]$.
We done with $j = 4$.
$j = 5 \Rightarrow r = 0 \Rightarrow$ rhs = $\mu_1 \mu_2^2 Q_3(5h) = \mu_1 \mu_2^2 [A_1 A_2^2 + A_2 A_1 A_2 + A_2^2 A_1]$.
We done with $j = 5$.
$j = 6 \Rightarrow r = 0 \Rightarrow$ rhs = $\mu_2^3 Q_3(6h) = \mu_2^3 A_2^3$.
Now, apply the evaluation procedure to $k = 3$, for all the contingencies, to get

$$\begin{aligned} \text{rhs} = {} & \mu_0^3 A_0^3 + \mu_0^2 \mu_1 [A_0^2 A_1 + A_0 A_1 A_0 + A_1 A_0^2] + \mu_0 \mu_1^2 [A_0 A_1^2 + A_1 A_0 A_1 + A_1^2 A_0] \\ & + \mu_0^2 \mu_2 [A_0^2 A_2 + A_2 A_0^2 + A_0 A_2 A_0] + \mu_1^3 A_1^3 \\ & + \mu_0 \mu_1 \mu_2 \begin{bmatrix} A_0 A_1 A_2 + A_0 A_2 A_1 + A_1 A_0 A_2 \\ + A_1 A_2 A_0 + A_2 A_0 A_1 + A_2 A_1 A_0 \end{bmatrix} \\ & + \mu_1^2 \mu_2 [A_1^2 A_2 + A_1 A_2 A_1 + A_2 A_1^2] + \mu_0 \mu_2^2 [A_0 A_2^2 + A_2 A_0 A_2 + A_2^2 A_0] \\ & + \mu_1 \mu_2^2 [A_1 A_2^2 + A_2 A_1 A_2 + A_2^2 A_1] + \mu_3^3 A_2^3 \end{aligned}$$

$$= \left(\sum_{i=0}^{2} \mu_i A_i \right)^3 = \text{lhs}$$

So the theorem is also true for $k = 3$ and hence true for $k \in \{0,1,2,3\}$.
Now we can apply the induction principle to k. Assume that the lemma is valid for $4 \le k \le n$, for some integer n. Then

$$\left(\sum_{i=0}^{2} \mu_i A_i \right)^{n+1} = (\mu_0 A_0 + \mu A_1 + \mu_2 A_2) \left(\sum_{i=0}^{2} \mu_i A_i \right)^n$$

$$= [(\mu_0 A_0 + \mu_1 A_1 + \mu_2 A_2)] \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n(jh) \big| \mu = 0,$$

in all permutation terms in $\mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n}$, involving $A_0^{r_0} A_1^{r_1} A_2^{r_2}$, for which $(r_0, r_1, r_2) \ne (r, 2n - j - 2r, r + j - n)$, (by the induction hypothesis)

We now examine $\mu_0 A_0 \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n\left(jh\right)$;

$$\sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n(jh) = \sum_{j=0}^{2n} \sum_{r=1}^{\left[\left[\frac{2n-j}{2}\right]\right]+1} \mu_0^{r-1} \mu_1^{2n-j-2(r-1)} \mu_2^{r-1+j-n} Q_n(jh)$$

$$= \sum_{j=0}^{2n} \sum_{r=1}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^{r-1} \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1} Q_n(jh) = \sum_{j=0}^{2(n+1)} \sum_{r=1}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^{r-1} \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} Q_n(jh)$$

(since $Q_n([2(n+1)h]) = 0$, $Q_n([2n+1)h]) = 0$ by (i) of lemma 2.6 of [8]

$$= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^{r-1} \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} Q_n(jh)$$

(since$r = 0$ is infeasible and so may be discarded)

Hence $\mu_0 A_0 \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2(n-j)}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n(jh)$

$$= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^r \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} A_0 Q_n\left(jh\right) \tag{2.5}$$

Now we examine $\mu_1 A_1 \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n\left(jh\right)$;

$$\sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n(jh) = \sum_{j=1}^{2n+1} \sum_{r=0}^{\left[\left[\frac{2n-(j-1)}{2}\right]\right]} \mu_0^r \mu_1^{2n-(j-1)-2r} \mu_2^{r+j-1-n} Q_n([j-1]h)$$

$$= \sum_{j=1}^{2n+1} \sum_{r=0}^{\left[\left[\frac{2n+1-j)}{2}\right]\right]} \mu_0^r \mu_1^{2n+1-j-2r} \mu_2^{r+j-(n+1)} Q_n([j-1]h) = \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j)}{2}\right]\right]} \mu_0^r \mu_1^{2n+1-j-2r} \mu_2^{r+j-(n+1)} Q_n([j-1]h),$$

$$= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j)}{2}\right]\right]} \mu_0^r \mu_1^{2n+1-j-2r} \mu_2^{r+j-(n+1)} Q_n([j-1]h),$$

since $Q_n([2n+2-1]h) = Q_n([2n+1]h]) = 0$, by (i), lemma 2.6 of [8]
andQ_n $([0-1]h]) = 0$, by (iii), lemma 2.5 of [8]
Hence

$\mu_1 A_1 \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n(jh)$

$$= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j)}{2}\right]\right]} \mu_0^r \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} A_1 Q_n\left([j-1]h\right) \tag{2.6}$$

Finally we examine $\mu_2 A_2 \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n(jh)$;

$$\sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n(jh) = \sum_{j=2}^{2n+2} \sum_{r=0}^{\left[\left[\frac{2n-(j-2)}{2}\right]\right]} \mu_0^r \mu_1^{2n-(j-2)-2r} \mu_2^{r+j-2-n} A_1 Q_n([j-2]h)$$

$$= \sum_{j=2}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2n-(j-2)}{2}\right]\right]} \mu_0^r \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)-1} A_1 Q_n([j-2]h)$$

$$= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^r \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)-1} A_1 Q_n([j-2]h)$$

since $Q_n([0-2]h) = Q_n(-2h) = 0$ and $Q_n([1-2]h) = Q_n(-h) = 0$, by lemma 2.5of [8].

Hence $\mu_2 A_2 \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r \mu_1^{2n-j-2r} \mu_2^{r+j-n} Q_n(jh)$

$$= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^r \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} A_2 Q_n([j-2]h) \qquad (2.7)$$

Now add up expressions (2.5), (2.6) and (2.7) to obtain

$$\sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^r \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} [A_0 Q_n(jh) + A_1 Q_n([j-1]h) + A_2 Q_n([j-2]h)$$

However, $Q_{n+1}(jh) = A_0 Q_n(jh) + A_1 Q_n([j-1]h) + A_2 Q_n([j-2]h)$, from the determining equation (1.7), yielding

Expressions (2.5) + (2.6) + (2.7) $= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^r \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} Q_{n+1}(jh)$

Hence,

$$\left(\sum_{i=0}^{2} \mu_i A_i\right)^{n+1} = \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left(\left[\frac{2(n+1)-j}{2}\right]\right)} \mu_0^r \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} Q_{n+1}(jh) \Big| \mu = 0$$

in all permutation terms in $\mu_0^r,\ \mu_1^{2(n+1)-j-2r} \mu_0^{r+j-(n+1)}$, involving $A_0^{r_0} A_1^{r_1} A_2^{r_2}$, for which $(r_0, r_1, r_2) \neq (r, 2(n+1)-j-2r, r+j-(n+1))$, where $\mu = (\mu_0, \mu_1, \mu_2)$.

So the theorem is true for $k = n + 1$ and hence true for every nonnegative integer k.

2.4 Theorem Indirectly Relating $\left(\sum_{i=0}^{2} \mu_i A_i\right)^k$ to $Q_k(jh)$ in a More Computationally Efficient Form.

For any real variables, μ_0, μ_1, μ_2 and for any integer $k \geq 0$

$$\left(\sum_{i=0}^{2}\mu_i A_i\right)^k = \begin{cases} \sum_{j=0}^{2k}\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \mu_0^r \mu_1^{2k-j-2r}\mu_2^{r+j-k} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k}, \text{ if } k\geq 1 \\ I_n, \text{if } k=0 \end{cases} \quad (2.8)$$

for all feasible (nonnegative integer) superscripts.

Proof

$k = 1 \Longrightarrow$ lhs = $\mu_0 A_0 + \mu_1 A_1 + \mu_2 A_2$; for the rhs, $k = 1 \Longrightarrow j \in \{0, 1, 2\}$

$j = 0 \Longrightarrow$ r ∈ {0, 1}; $j = 0$, r = 0 ⟹ infeasibility; $j = 0$, r = 1 ⟹ rhs = $\mu_0 A_0$

$j = 1 \Longrightarrow$ r = 0 ⟹ rhs = $\mu_1 A_1$; $j = 2 \Longrightarrow$ r = 0 ⟹ rhs = $\mu_2 A_2$

Adding up above contingencies for $k = 1$, we get rhs = $\mu_0 A_0 + \mu_1 A_1 + \mu_2 A_2$ = lhs

$k = 2 \Longrightarrow j \in \{0, 1, 2, 3, 4\}$; $j = 0 \Longrightarrow r \in \{0, 1, 2\}$; $j = 0, r \in \{0, 1\} \Longrightarrow$ infeasibility

$j = 0$, r = 2 ⟹ rhs = $\mu_0 A_0^2$; $j = 1 \Longrightarrow r \in \{0, 1\}$; $j = 1$, $r = 0 \Longrightarrow$ infeasibility

$j = 1$, r = 1 ⟹ rhs= $\mu_0\mu_1[A_0A_1 + A_1A_0]$; $j = 2 \Longrightarrow r \in \{0, 1\}$

$j = 2$, r = 0 ⟹ rhs = $\mu_1^2 A_1^2$; $j = 2$, $r = 1 \Longrightarrow$ rhs = $\mu_0\mu_2[A_0A_2 + A_2A_0]$.

$j = 3$, ⟹ r = 0 ⟹ rhs = $\mu_1\mu_2[A_1A_2 + A_2A_1]$; $j = 4$, ⟹ r = 0 ⟹ rhs = $\mu_2^2 A_2^2$

Adding up the rhs for $k = 2$, we get

$$rhs = \mu_0^2A_0^2 + \mu_0\mu_1[A_0A_1 + A_1A_0] + \mu_1^2A_1^2 + \mu_1\mu_2[A_1A_2 + A_2A_1] + \mu_2^2A_2^2 + \mu_0\mu_2[A_0A_2 + A_2A_0] = lhs$$

So the result is true for $k \in \{1, 2\}$

The rest of the proof is by mathematical induction.

Assume that the lemma is true for $3 \leq k \leq n$, for some integer n. Then

$$\left(\sum_{i=0}^{2}\mu_i A_i\right)^{n+1} = (\mu_0A_0 + \mu_1A_1 + \mu_2A_2)\left[\sum_{i=0}^{2}\mu_i A_i\right]^n$$

$$\left(\sum_{i=0}^{2}\mu_i A_i\right)^n = \sum_{j=0}^{2n}\sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r\mu_1^{2n-j-2r}\mu_2^{r+j-n}\sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n}$$

We proceed to examine

$$\mu_0A_0\sum_{j=0}^{2n}\sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r\mu_1^{2n-j-2r}\mu_2^{r+j-n}\sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n}$$

Now, $$\sum_{j=0}^{2n}\sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^r\mu_1^{2n-j-2r}\mu_2^{r+j-n}\sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n}$$

$$= \sum_{j=0}^{2n}\sum_{r=1}^{\left[\left[\frac{2n-j}{2}\right]\right]+1} \mu_0^{r-1}\mu_1^{2n-j-2(r-1)}\mu_2^{r-1+j-n}\sum_{(v_1,\cdots,v_n)\in P_{0(r-1),1(2n-j-2(r-1)),2(r-1+j-n)}} A_{v_1}\cdots A_{v_n}$$

$$= \sum_{j=0}^{2n}\sum_{r=1}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^{r-1}\mu_1^{2(n+1)-j-2r}\mu_2^{r+j-(n+1)}\sum_{(v_1,\cdots,v_n)\in P_{0(r-1),1(2(n+1)-j-2r),2(r+j-(n+1))}} A_{v_1}\cdots A_{v_n}$$

$$= \sum_{j=0}^{2(n+1)}\sum_{r=1}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^{r-1}\mu_1^{2(n+1)-j-2r}\mu_2^{r+j-(n+1)}\sum_{(v_1,\cdots,v_n)\in P_{0(r-1),1(2(n+1)-j-2r),2(r+j-(n+1))}} A_{v_1}\cdots A_{v_n}$$

since $j \in \{2n + 1, 2(n+1)\}$, $r \geq 1 \Longrightarrow 2(n+1) - j - 2r < 0$ and so, the terms with $j \in \{2n + 1, 2(n+1)\}$ drop out.

The last expression is the same as:

$$\sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^{r-1} \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} \sum_{(v_1,\cdots,v_n)\in P_{0(r-1),1(2(n+1)-j-2r),2(r+j-(n+1))}} A_{v_1}\cdots A_{v_n},$$

since $r=0 \Rightarrow r-1=-1<0,$ so that the terms with r = 0 drop out.
Hence

$$\mu_0 A_0 \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^{r} \mu_1^{2n-j-2r} \mu_2^{r+j-n} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n}$$

$$= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^{r} \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2(n+1)-j-2r),2(r+j-(n+1))}} A_{v_1}\cdots A_{v_{n+1}}, \qquad (2.9)$$

with a leading A_0.

Next, we examine $\mu_1 A_1 \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^{r} \mu_1^{2n-j-2r} \mu_2^{r+j-n} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n}$

Clearly, $\sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^{r} \mu_1^{2n-j-2r} \mu_0^{r+j-n} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n}$

$$= \sum_{j=1}^{2n+1} \sum_{r=0}^{\left[\left[\frac{2n-(j-1)}{2}\right]\right]} \mu_0^{r} \mu_1^{2n-(j-1)-2r} \mu_2^{r+j-1-n} \sum_{(v_1,\cdots,v_n)\in P_{0(r-1),1(2n-(j-1)-2r),2(r+j-1-n)}} A_{v_1}\cdots A_{v_n}$$

$$= \sum_{j=0}^{2n+1} \sum_{r=0}^{\left[\left[\frac{2n+1-j}{2}\right]\right]} \mu_0^{r} \mu_1^{2n+1-j-2r} \mu_2^{r+j-(n+1)} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n+1-j-2r),2(r+j-(n+1))}} A_{v_1}\cdots A_{v_{n+1}},$$

$$= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^{r} \mu_1^{2n+1-j-2r} \mu_2^{r+j-(n+1)} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2n+1-j-2r),2(r+j-(n+1))}} A_{v_1}\cdots A_{v_n},$$

since $j=0 \Rightarrow r<n+1 \Rightarrow r+j-(n+1)<0;\ j=2(n+1) \Rightarrow 2n+1-j=-1<0.$ Therefore, theterms with$j$ = 0 and $j=2(n+1)$ drop out. Hence

$$\mu_1 A_1 \sum_{j=0}^{2n} \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \mu_0^{r} \mu_1^{2n-j-2r} \mu_2^{r+j-n} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n}$$

$$= \sum_{j=0}^{2(n+1)} \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \mu_0^{r} \mu_1^{2(n+1)-j-2r} \mu_2^{r+j-(n+1)} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2(n+1)-j-2r),2(r+j-(n+1))}} A_{v_1}\cdots A_{v_{n+1}}, \qquad (2.10)$$

with a leading A_1.

Finally, $\sum_{j=0}^{2n}\sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]}\mu_0^r\mu_1^{2n-j-2r}\mu_0^{r+j-n}\sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}}A_{v_1}\cdots A_{v_n}$

$$=\sum_{j=2}^{2(n+1)}\sum_{r=0}^{\left[\left[\frac{2n-(j-2)}{2}\right]\right]}\mu_0^r\mu_1^{2n-(j-2)-2r}\mu_2^{r+j-2-n}\sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-[j-2]-2r),2(r+[j-2]-n)}}A_{v_1}\cdots A_{v_n}$$

$$=\sum_{j=2}^{2(n+1)}\sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]}\mu_0^r\mu_1^{2(n+1)-j-2r}\mu_2^{r+j-(n+1)-1}\sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2(n+1)-j-2r),2(r+j-[n+1]-1)}}A_{v_1}\cdots A_{v_{n+1}},$$

$$=\sum_{j=0}^{2(n+1)}\sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]}\mu_0^r\mu_1^{2(n+1)-j-2r}\mu_2^{r+j-(n+1)-1}\sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2(n+1)-j-2r),2(r+j-[n+1]-1)}}A_{v_1}\cdots A_{v_{n+1}},$$

since $j=0\Rightarrow r\le n+1\Rightarrow r+j-(n+1)-1\le -1<0$

and $j=1\Rightarrow r\le n\Rightarrow r+j-(n+1)-1\le -1<0$. Hence the terms with $j\in\{1,2\}$ drop out.

Therefore, $\mu_2A_2\sum_{j=0}^{2n}\sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]}\mu_0^r\mu_1^{2n-j-2r}\mu_0^{r+j-n}\sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}}A_{v_1}\cdots A_{v_n}$

$$=\sum_{j=0}^{2(n+1)}\sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]}\mu_0^r\mu_1^{2(n+1)-j-2r}\mu_2^{r+j-(n+1)}\sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2(n+1)-j-2r),2(r+j-(n+1))}}A_{v_1}\cdots A_{v_{n+1}},\quad(2.11)$$

with a leading A_2.

Add up the terms with leading A_0, A_1 and A_2 respectively to get

$$\left(\sum_{i=0}^{2}\mu_iA_i\right)^{n+1}=\sum_{j=0}^{2(n+1)}\sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]}\mu_0^r\mu_1^{2(n+1)-j-2r}\mu_2^{r+j-(n+1)}\sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2(n+1)-j-2r),2(r+j-(n+1))}}A_{v_1}\cdots A_{v_{n+1}},(2.12)$$

leading to the conclusion that the theorem is true for $k = n + 1$ and hence true for every nonnegative integer k. This completes the proof.

See section lemma 2.4 of [8] for further explanation on 'leading' and 'trailing' permutation objects.

Observe that the above theorem is independent of the expression for $Q_k(jh)$; the j above is justa dummy variable; we could just as well have used $\tilde{j}$ and $\tilde{r}$, in place of j and r respectively.

3.0 Results and Discussions

3.1 First Corollary to Theorem 2.4: Expressing $Q_k(jh)$ as A Sum of Partial Derivatives of $\left(\sum_{i=0}^{2}\mu_iA_i\right)^k$

For any nonnegative integers $j,k: j+k\neq 0,\ j\ge k-r,$

$$Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \frac{1}{r!(2k-j-2r)!(r+j-k)!} \frac{\partial^k}{\partial\mu_0^r \partial\mu_1^{2k-j-2r} \partial\mu_2^{r+j-k}} \left(\sum_{i=0}^{2} \mu_i A_i\right)^k \qquad (3.1)$$

Proof

From the preceding result (theorem 2.4), we deduce immediately that for fixed r, j and $k,$

$$\frac{\partial^k}{\partial\mu_0^r \partial\mu_1^{2k-j-2r} \partial\mu_2^{r+j-k}} \left(\sum_{i=0}^{2} \mu_i A_i\right)^k = r!(2k-j-2r)!(r+j-k)! \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k} \qquad (3.2)$$

Equivalently, $\displaystyle\frac{1}{r!(2k-j-2r)!(r+j-k)!} \frac{\partial^k}{\partial\mu_0^r \partial\mu_1^{2k-j-2r} \partial\mu_2^{r+j-k}} \left(\sum_{i=0}^{2} \mu_i A_i\right)^k$

$$= \sum_{(v_1,v_2,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}A_{v_2}\cdots A_{v_k} \qquad (3.3)$$

provided $j \ge k - r.$

Hence, for fixed k and j and for r ranging from 0 to $\left[\left[\frac{2k-j}{2}\right]\right],\ j \ge k-r,$, we obtain

$$\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \frac{1}{r!(2k-j-2r)!(r+j-k)!} \frac{\partial^k}{\partial\mu_0^r \partial\mu_1^{2k-j-2r} \partial\mu_2^{r+j-k}} \left(\sum_{i=0}^{2} \mu_i A_i\right)^k \qquad (3.4)$$

$$= \sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k} = Q_k(jh), \text{ (by theorem 3.2 of [8]).}$$

This completes the proof of the corollary.
Also, in view of theorems 3.1 and 3.2of [8], we can restate the above corollary in the equivalent forms:

3.2 Second Corollary to thm. 2.4: $Q_k(jh)$ in three piece-wise sums of partials of $\left(\sum_{i=0}^{2} \mu_i A_i\right)^k$

For any nonnegative integers j and $k,$

$$
Q_k(jh)=\begin{cases}\displaystyle\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]}\frac{1}{r!(2k-j-2r)!(r+j-k)!}\frac{\partial^k}{\partial\mu_0^r\partial\mu_1^{2k-j-2r}\partial\mu_2^{r+j-k}}\left(\sum_{i=0}^{2}\mu_iA_i\right)^k;\ 1\le k\le j\\[2ex]
\displaystyle\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\frac{1}{(r+j-k)!(j-2r)!r!}\frac{\partial^k}{\partial\mu_0^{r+k-j}\partial\mu_1^{j-2r}\partial\mu_2^{r}}\left(\sum_{i=0}^{2}\mu_iA_i\right)^k,\text{if }\ 0\le j\le k\\[2ex]
0,\text{ if }\ j\ge 2k+1,\text{ or if }\ (A_2=0\text{ and }\ j>k).\end{cases}\qquad(3.5)
$$

3.3 Third Corollary to thm. 2.4: $Q_k(jh)$ in implicit piece-wise composite sum of partials form

For any nonnegative integers j and k,

$$
\begin{aligned}
Q_k(jh)\\
=&\left[\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]}\frac{1}{r!(2k-j-2r)!(r+j-k)!}\frac{\partial^k}{\partial\mu_0^r\partial\mu_1^{2k-j-2r}\partial\mu_2^{r+j-k}}\left(\sum_{i=0}^{2}\mu_iA_i\right)^k\right]\operatorname{sgn}(\max\{0,j+1-k\})\\
&+\left[\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\frac{1}{(r+j-k)!(j-2r)!r!}\frac{\partial^k}{\partial\mu_0^{r+k-j}\partial\mu_1^{j-2r}\partial\mu_2^{r}}\left(\sum_{i=0}^{2}\mu_iA_i\right)^k\right]\operatorname{sgn}(\max\{0,k-j\})\qquad(3.6)
\end{aligned}
$$

Proof

The corollary follows by noting the following facts:

(i) if $j=k, \operatorname{sgn}(\max\{0,k-j\})=0,$ and $\operatorname{sgn}(\max\{0,j+1-k\})=1.$ Then the expression for $Q_k(jh)$ coincides with those of theorems 3.1 and 3.2 of [8], for $j=k$,in view of corollary 3.1.

(ii) if $0\le j<k,$ then $j+1-k\le 0$ and $k-j>0.$ Then the expression for $Q_k(jh)$ coincides with that of theorem 3.1 of [8], as the first component summations drop out.

(iii) if $0\le k<j,$ then $j+1-k>0$ and $k-j<0.$ Then the expression for $Q_k(jh)$ coincides with that of theorem 3.2of [8], as the second component summations drop out.

4.0 Conclusion

The results in this article attest to the fact that we have extended the previous single-delay result by[9], together with appropriate embellishments through the unfolding of intricate inter–play of the greatest integer function and the permutation objects.By using the permutation fact sheets in lemma 2.4 of [8], the greatest integer function analysis, change of variables technique and deft application of mathematical induction principles we were able to relate $\left(\sum_{i=0}^{2}\mu_iA_i\right)^k$ to permutation objects; then we appropriated this relationship to develop new structures for the determining matrices, involvingcertain partial derivatives of $\left(\sum_{i=0}^{2}\mu_iA_i\right)^k$ and permutations of 0, 1 and 2, which would pave the way for the establishing of equality of ranks of controllability matrices for finite and infinite horizons and hence the computational investigation of Euclidean controllability with respect to the double–delay control model.The mathematical icing on the cake was our deft application of the max and sgn functions and their composite function,sgn (max {.,.})in combination with the afore-mentioned partial derivatives, to obtain alternative expressions for determining matrices. Such applications are optimal, in the sense that they obviate the need for explicit piece–wise representations of those and many other discrete mathematical objects.

References

[1] Chidume, C. An Introduction to Metric Spaces. The Abdus Salam, International Centre for Theoretical Physics, Trieste, Italy, 2003.
[2] Chidume, C. Applicable Functional Analysis. The Abdus Salam, International Centre for Theoretical Physics, Trieste, Italy, 2007.
[3] Royden, H.L. Real Analysis, 3rd Ed. Macmillan Publishing Co., New York, 1988.
[4] Chukwu, E. N. (1992). Stability and Time-optimal control of hereditary systems. Academic Press, New York, 1992.
[5] Hale, J. K. Theory of functional differential equations. Applied Mathematical Science, 1977, Vol. 3, Springer-Verlag, New York.
[6] Tadmore, G. Functional differential equations of retarded and neutral types: Analytical solutions and piecewise continuous controls. J. Differential equations, 1984, Vol. 51, No. 2, Pp. 151-181.
[7] Gabasov, R. and Kirillova, F. The qualitative theory of optimal processes. MarcelDekker Inc., New York, 1976.
[8] Ukwu, C. (2014g). The structure of determining matrices for a class of double – delay control systems. International Journal of Mathematics and Statistics Invention (IJMSI), Volume 2, Issue 2, February 2014, Pp. 14-31.
[9] Ukwu, C. Euclidean Controllability and Cores of Euclidean Targets for Differential difference systems, Master of Science Thesis in Applied Mathematics with Operations Research,(Unpublished), North Carolina State University, Raleigh, N. C.U.S.A., 1992