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Stability of FTCS Scheme for the Solution of Advection equation

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Abstract

The forward in time and central in space scheme is used for the solution of one dimensional partial differential equation. On the application of the Von Neumann stability analysis, we find that, the forward in time and central in space scheme is not stable for all values of α .

Keywords: Advection equation, Courant number, Taylor series, Numerical stability, Forward in time and central in space scheme, Von Neumann stability analysis

1.0 Introduction

Let consider the forward in time and central in space (FTCS) scheme given as

$$u_j^{n+1} = u_j^n - \frac{\alpha\left(u_{j+1}^n - u_{j-1}^n\right)}{2} \quad (1)$$

where α is some constant [1]

We also consider the advection equation

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \quad (2)$$

Where, at

t=0, u(x,0)=f(x), 0≤x≤1

as x=0, u(0,t)=g(x), t≥0 (3)

Here, u=u(x,t) is an unknown function of x and t respectively. The unknown $u(x,t) = u(x_1, x_2, \dots, x_n, t)$ is either a scalar or a vector function [2].

The solution of (2) and (3) is gotten in an arbitrary region $\Re \times [0,T]$ with suitable boundary condition $\partial\Re \times [0,T]$, where ∂R is the boundary of $\Re$ [3].

Using the FTCS or forward in time and central in space scheme in (1) as

$$u_j^{n+1} = u_j^n - \frac{\alpha\left(u_{j+1}^n - u_{j-1}^n\right)}{2}$$

and applying Taylor series expansion [4], scheme (1) becomes

$$u_j^{n+1} = u_j^n + \Delta t u_t^n + \frac{(\Delta t)^2}{2}u_{tt}^n + \cdots$$

$$u_{j+1}^n = u_j^n + \Delta x u_x^n + \frac{(\Delta x)^2}{2}u_{xx}^n + \cdots$$

$$u_{j-1}^n = u_j^n - \Delta x u_x^n + \frac{(\Delta x)^2}{2}u_{xx}^n - \cdots$$

$$u_j^n + \Delta t u_t^n + \frac{(\Delta t)^2}{2}u_{tt}^n = u_j^n - \alpha(u_j^n + \Delta x u_x^n + \frac{(\Delta x)^2}{2}u_{xx}^n - u_j^n + \Delta x u_x^n - \frac{(\Delta x)^2}{2}u_{xx}^n$$

$$= u_j^n - \alpha \Delta x u_x^n (4)$$

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$$u_j^n - u_j^n + \Delta t u_t^n + \alpha \Delta x u_x^n + \frac{(\Delta t)^2}{2} u_{tt}^n$$
$$= \Delta t u_t^n + \alpha \Delta x u_x^n$$
$$-\Delta t u_t^n = \alpha \Delta x u_x^n \qquad (5)$$
$$u_t^n = \frac{\alpha \Delta x}{\Delta t} u_x^n (6)$$

The quantity $\frac{\alpha \Delta x}{\Delta t}$ is called the courant number [5].
We now see that the forward in time and central in space scheme satisfies the advection equation.
We now give a general result due to Von Neumann which deal on the stability of the numerical scheme.
Theorem: (Von Neumann [6])
The error distribution ε(x,t) at any time and spatial location can be decomposed into a Fourier series of the form

$$\varepsilon(x,t) = \sum_m b_m (t) e^{ik_m x},$$

Where $i = \sqrt{-1}$ and k_m is the wave number when the physical domain extending from $x = 0\ tox = 1$ has periodic boundaries and the above series has M+1 wave components, where
$M = \frac{L}{\Delta x}$ and the wave number k_m is written as $k_m = \frac{M\pi}{L}$, M=0,1,2,…,m

2.0 Main Result

Let us consider the FTCS scheme given by equation (1) which is reproduced below

$$u_i^{n+1} = u_i^n - \frac{\alpha(u_{i+1}^n - u_{i-1}^n)}{2}$$

Since the computed solution, N_i must satisfy the discretized equation,
we have,

$$\frac{N_i^{n+1} - N_i^n}{\Delta t} + \frac{b(N_{i+1}^n - N_{i-1}^n)}{2\Delta x} = 0 \qquad (7)$$

Substituting$N_i^n = D_i^n + \varepsilon_i^n$ in equation (7), we obtain

$$\frac{[(D_i^{n+1} + \varepsilon_i^{n+1}) - (D_i^n - \varepsilon_i^n)]}{\Delta t} + \frac{b[(D_{i+1}^n + \varepsilon_{i+1}^n) - (D_{i-1}^n - \varepsilon_{i-1}^n)]}{2\Delta x} = 0$$

Or

$$\frac{D_i^{n+1} - D_i^n}{\Delta t} + \frac{b(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)}{2\Delta x} = 0 \qquad (8)$$

Since D is the exact solution of the discretized equation, the first two terms in equation (8) taken together are exactly equal to zero and equation (8) reduces to

$$\frac{(\varepsilon_i^{n+1} - \varepsilon_i^n)}{\Delta t} + \frac{b(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n)}{2\Delta x} = 0 \qquad (9)$$

Which is the error solution equation for the FTCS scheme.
Substituting

$$\varepsilon_m(x,t) = e^{h_m t} e^{ik_m x}$$

Or

$$\varepsilon_i^n = e^{h_m n \Delta t} e^{ik_m i \Delta x} \qquad (10)$$

Dividing equation (9) by$e^{lmt} e^{ik_m t}, weobtain$

$(e^{l_m \Delta t} - 1) + \frac{b\Delta t}{2\Delta x}\left(e^{ik_m \Delta x} - e^{-ik_m \Delta x}\right) = 0$ (11)

Denoting $\frac{e_{m,1}^{n+1}}{e_{n,1}^{n}} = G$

$= e^{b_m \Delta t}$, and noting that

$(e^{ik_m} - e^{-ik_m \Delta x}) = 2isin(k_m \Delta x),$

Equation (11) can be written as

$G = 1 - i\alpha sin\emptyset(12)$

where

$\emptyset = k_m \Delta x$

Evaluating the modulus of G, we note that

$|G| = |1 - i\alpha sin\emptyset| > 0$ (13)

3.0 Conclusion:

This implies that, the FTCS scheme for one dimensional wave equation is unstable for all values of α. The unstable condition arises due to the amplification factor which is decomposed into its real and imaginary parts.

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