Journal of the Nigerian Association of Mathematical Physics
Volume27, (*July, 2014*), **pp 37 – 44**
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One Step Predictor Corrector Method for the Solution of Stiff $y' = f(x, y)$

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Abstract

In this paper attention is directed towards the construction of one step, three hybrid points constant predictor corrector method for the solution of stiff $y' = f(x, y)$. ***We adopted the method of collocation and interpolation of power series and exponential function approximate solution to generate continuous linear multistep method which was evaluated at some selected grid points to give discrete linear multistep method. The method was implemented using a constant order predictor of order five over an overlapping interval. The basic properties of the derived corrector was investigated and found to be zero stable, consistent and convergent. The method was tested on some numerical examples and found to compare favourably with the existing methods.***

Keywords: Interpolation, collocation, grid points, consistent, zero stable, convergent, block method, stiff, corrector.
2010 **AMS** Subject Classification: 65L05, 65L06, 65D30

1.0 Introduction

A very important special class of differential equations taken up in the initial value problems termed as stiff differential equation result from the phenomenon with widely differing time scale. There is no universally accepted definition of stiffness. Stiffness is a subtle, difficult and important concept in the numerical solution of ordinary differential equations.

The initial value problems with stiff first order ordinary differential equation systems occur in many fields of engineering science, particularly in the studies of electrical circuits, vibrations, chemical reactions and so on. Stiff differential equations are ubiquitous in astrochemical kinetics, many control systems and electronics, but also in many non-industrial area like weather prediction and biology.

This paper considered the numerical method for solving stiff first order initial value problems of ordinary differential equations of the form

$$y' = f(x, y), y(x_0) = y_0 \qquad (1)$$

where x_0 is the initial point, y_0 is the solution at the initial point. f is assumed to be continuous and satisfies the lipschitz theorem for the existence and uniqueness of solution.

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Definition 1 [1]: *The initial value problem (1) is considered to be stiff oscillatory if the eigenvalues* $\{\lambda_j = u_j + iv_j, j = 1(1)m\}$ *of the Jacobian* $J = \frac{\partial f}{\partial y}$ *possess the following properties* $u_j < 0, j = 1(1)m, \underset{1\le j\le m}{Max}|u_j| > \underset{i\le j\le m}{Max}|u_j|$ *or if the stiffness ratio satisfies* $S = \underset{i,j}{Max}\left|\frac{u_i}{u_j}\right| > 1$ *and* $|u_i| < |u_j|$ *for at leastpair of* j *in* $1 \le j \le m.$

Most of the conventional numerical solver cannot efficiently cope with stiff problems because they lack the stability characteristics [2]. Most of the methodsproposed for the solution of stiff problems are numerically unstable unless thestep sizes are taken to be extremely small. Scholars have reported that theadoptions of implicit schemes which are A-stable method are better for the solution of stiff problems [3, 4].

Definition 2 [5]: *A numerical method is said to be A-stable if the whole of the left-half plane* $\{z : \mathrm{Re}(z) \le 0\}$ *is contained in the region* $\{z : \mathrm{Re}(z) \le 1\},$ *where* $R(z)$ *is called the stability polynomial of the method.*

Scholars have proposed different numerical method for the solution of (1) by adopting different approximate polynomial ranging from backward differentiation formula, power series polynomial, Fourier series Polynomial, Langrange Polynomial to mention few. Though, the choice of the approximate solutiondepends largely on the type of problem to be solved, not withstanding, mostof these methods do not give good stability properties hence they fail when the problems is stiff or oscillatory. The introduction of step method has help greatly in the solution of stiff problem because most of these problems givebetter stability condition and have circumvented the Dalquist stability barrier [6].

Different method of implementation have been proposed by scholars ranging frompredictor-corrector method to block method. Block method has been reportedin literature to be better than the predictor-corrector method in terms of costof development, time of execution and accuracy.Block method was proposed to take care ofsome of the setbacks of the predictor-corrector method [7-13].

In quest for the method that gives better stability condition, scholars proposed an approximate solution which combined power series polynomial andexponential function [3, 14]. It was discovered that this method gives an A-stablemethod no matter how the grid points are selected. Therefore we seek to address this setback by proposing a method that shares the properties of both the block method and the predictor-corrector method using the approximatemethod proposed by [3, 14].

2.0 Methodology

2.1 Derivation of Corrector

We consider a combination of power series and exponential function approximate solution of the form.

$$y(x) = \sum_{j=0}^{r+s-2} a_j x^j + a_{r+s-1} e^{\alpha x} \qquad (2)$$

where r and s are the numbers of interpolation and collocation points respectively. x^j is the polynomial basis function, $a_j\text{'s} \in \Re$ are constants to be determined.

Substituting the first derivative of (2) into (1) gives

$$f(x, y) = \sum_{j=1}^{r+s-2} j a_j x^{j-1} + \alpha a_{r+s-1} e^{\alpha x} \qquad (3)$$

Collocating (3) at x_{n+r} $r = 0(\frac{1}{4})1$ and interpolating (2) at $x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{2}}$ gives the system of non-linear equation of the form

$$AX = U \qquad (4)$$

Where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7]^T$$

$$U = \left[y_n, y_{n+\frac{1}{4}}, y_{n+\frac{1}{2}}, f_n, f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1} \right]^T$$

and

$$X = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \left(1+\alpha x_n + \frac{\alpha_n^2 x_n^2}{2!} + \frac{\alpha_n^3 x_n^3}{3!} + \frac{\alpha_n^4 x_n^4}{4!} + \frac{\alpha_n^5 x_n^5}{5!} + \frac{\alpha_n^6 x_n^6}{6!} + \frac{\alpha_n^7 x_n^7}{7!}\right) \\
1 & x_{n+\frac{1}{4}} & x^2_{n+\frac{1}{4}} & x^3_{n+\frac{1}{4}} & x^4_{n+\frac{1}{4}} & x^5_{n+\frac{1}{4}} & x^6_{n+\frac{1}{4}} & \left(1+\alpha x_{n+\frac{1}{4}} + \frac{\alpha^2_{n+\frac{1}{4}} x^2_{n+\frac{1}{4}}}{2!} + \frac{\alpha^3_{n+\frac{1}{4}} x^3_{n+\frac{1}{4}}}{3!} + \frac{\alpha^4_{n+\frac{1}{4}} x^4_{n+\frac{1}{4}}}{4!} + \frac{\alpha^5_{n+\frac{1}{4}} x^5_{n+\frac{1}{4}}}{5!} + \frac{\alpha^6_{n+\frac{1}{4}} x^6_{n+\frac{1}{4}}}{6!} + \frac{\alpha^7_{n+\frac{1}{4}} x^7_{n+\frac{1}{4}}}{7!}\right) \\
1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} & \left(1+\alpha x_{n+\frac{1}{2}} + \frac{\alpha^2_{n+\frac{1}{2}} x^2_{n+\frac{1}{2}}}{2!} + \frac{\alpha^3_{n+\frac{1}{2}} x^3_{n+\frac{1}{2}}}{3!} + \frac{\alpha^4_{n+\frac{1}{2}} x^4_{n+\frac{1}{2}}}{4!} + \frac{\alpha^5_{n+\frac{1}{2}} x^5_{n+\frac{1}{2}}}{5!} + \frac{\alpha^6_{n+\frac{1}{2}} x^6_{n+\frac{1}{2}}}{6!} + \frac{\alpha^7_{n+\frac{1}{2}} x^7_{n+\frac{1}{2}}}{7!}\right) \\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & \left(\alpha x_n + \frac{\alpha_n^3 x_n^2}{2!} + \frac{\alpha_n^4 x_n^3}{3!} + \frac{\alpha_n^5 x_n^4}{4!} + \frac{\alpha_n^6 x_n^5}{5!} + \frac{\alpha_n^7 x_n^6}{6!}\right) \\
0 & 1 & 2x_{n+\frac{1}{4}} & 3x^2_{n+\frac{1}{4}} & 4x^3_{n+\frac{1}{4}} & 5x^4_{n+\frac{1}{4}} & 6x^5_{n+\frac{1}{4}} & \left(\alpha x_{n+\frac{1}{4}} + \frac{\alpha^3_{n+\frac{1}{4}} x^2_{n+\frac{1}{4}}}{2!} + \frac{\alpha^4_{n+\frac{1}{4}} x^3_{n+\frac{1}{4}}}{3!} + \frac{\alpha^5_{n+\frac{1}{4}} x^4_{n+\frac{1}{4}}}{4!} + \frac{\alpha^6_{n+\frac{1}{4}} x^5_{n+\frac{1}{4}}}{5!} + \frac{\alpha^7_{n+\frac{1}{4}} x^6_{n+\frac{1}{4}}}{6!}\right) \\
0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} & \left(\alpha x_{n+\frac{1}{2}} + \frac{\alpha^3_{n+\frac{1}{2}} x^2_{n+\frac{1}{2}}}{2!} + \frac{\alpha^4_{n+\frac{1}{2}} x^3_{n+\frac{1}{2}}}{3!} + \frac{\alpha^5_{n+\frac{1}{2}} x^4_{n+\frac{1}{2}}}{4!} + \frac{\alpha^6_{n+\frac{1}{2}} x^5_{n+\frac{1}{2}}}{5!} + \frac{\alpha^7_{n+\frac{1}{2}} x^6_{n+\frac{1}{2}}}{6!}\right) \\
0 & 1 & 2x_{n+\frac{3}{4}} & 3x^2_{n+\frac{3}{4}} & 4x^3_{n+\frac{3}{4}} & 5x^4_{n+\frac{3}{4}} & 3x^5_{n+\frac{3}{4}} & \left(\alpha x_{n+\frac{3}{4}} + \frac{\alpha^3_{n+\frac{3}{4}} x^2_{n+\frac{3}{4}}}{2!} + \frac{\alpha^4_{n+\frac{3}{4}} x^3_{n+\frac{3}{4}}}{3!} + \frac{\alpha^5_{n+\frac{3}{4}} x^4_{n+\frac{3}{4}}}{4!} + \frac{\alpha^6_{n+\frac{3}{4}} x^5_{n+\frac{3}{4}}}{5!} + \frac{\alpha^7_{n+\frac{3}{4}} x^6_{n+\frac{3}{4}}}{6!}\right) \\
0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & \left(\alpha x_{n+1} + \frac{\alpha^3_{n+1} x^2_{n+1}}{2!} + \frac{\alpha^4_{n+1} x^3_{n+1}}{3!} + \frac{\alpha^5_{n+1} x^4_{n+1}}{4!} + \frac{\alpha^6_{n+1} x^5_{n+1}}{5!} + \frac{\alpha^7_{n+1} x^6_{n+1}}{6!}\right)
\end{bmatrix}$$

Solving (4) for $a_j's, j = 0(\frac{1}{4})1$ using Gaussian elimination method and substituting into (2) gives a continuous hybrid linear multistep method in the form

$$y(x) = \alpha_\mu(x) y_{n+\mu} + h\left(\sum_{j=0}^{1} \beta_j(x) f_{n+j} + \beta_k(x) f_{n+k}\right), k = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}. \qquad (5)$$

Where

$$\left.\begin{aligned}
\alpha_0 &= \frac{1}{17}(16896t^7 - 56576t^6 + 73632t^5 - 47040t^4 + 15096t^3 - 2052t^2 + 17) \\
\alpha_{\frac{1}{4}} &= \frac{64}{17}(384t^7 - 1088t^6 + 1080t^5 - 420t^4 + 34t^3 + 9t^2) \\
\alpha_{\frac{1}{2}} &= \frac{4}{17}(-10368t^7 + 31552t^6 - 35688t^5 + 18480t^4 - 4318t^3 + 369t^2) \\
\beta_0 &= \frac{1}{255}(17984t^2 - 61472t^6 + 82676t^5 - 55880t^4 + 20009t^3 - 3599t^2 + 255t) \\
\beta_{\frac{1}{4}} &= -\frac{4}{255}(-32896t^7 + 104448t^6 - 125704t^5 + 71340t^4 - 19006t^3 + 1881t^2) \\
\beta_{\frac{1}{2}} &= -\frac{2}{85}(-12544t^7 + 36992t^6 - 40176t^5 + 19840t^4 - 4454t^3 + 369t^2) \\
\beta_{\frac{3}{4}} &= \frac{4}{255}(-1664t^7 + 4352t^6 - 4136t^5 + 1820t^4 - 374t^3 + 29t^2) \\
\beta_1 &= -\frac{1}{255}(-704t^7 + 1632t^6 - 1436t^5 + 600t^4 - 119t^3 + 9t^2)
\end{aligned}\right\} \qquad (6)$$

$$t = \frac{x - x_n}{h}$$

Evaluating (5) at $t = 1$ gives our corrector

$$y_{n+1} + \frac{27}{17} y_n + \frac{64}{17} y_{n+\frac{1}{4}} - \frac{108}{17} y_{n+\frac{1}{2}} + h\left(\frac{9}{85} f_n + \frac{84}{85} f_{n+\frac{1}{4}} + \frac{54}{85} f_{n+\frac{1}{2}} - \frac{36}{85} f_{n+\frac{3}{4}} - \frac{6}{85} f_{n+1} \right) \qquad (7)$$

2.2 Derivation of Constant Order Predictor

The block method which we adopted as our constant order predictor was developed by Momoh et al [3]. They collocated (2) at $x_{n+s}, s = 0(\frac{1}{4})$ and interpolated (1) at x_n to obtain a discrete block method given as

$$A^{(0)} Y_m = e y_n + h\left[d f(y_n) + b F(Y_m) \right] \qquad (8)$$

where

$$Y_m = \begin{bmatrix} y_{n+\frac{1}{4}} & y_{n+\frac{1}{2}} & y_{n+\frac{3}{4}} & y_{n+1} \end{bmatrix}^T$$

$$F(Y_m) = \begin{bmatrix} f_{n+\frac{1}{4}} & f_{n+\frac{1}{2}} & f_{n+\frac{3}{4}} & f_{n+1} \end{bmatrix}^T$$

$$y_n = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_n \end{bmatrix}^T$$

$$f(y_n) = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_n \end{bmatrix}^T$$

$A^{(0)} = 4 \times 4$ identity matrix

$$e = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{2880} \\ 0 & 0 & 0 & \frac{29}{360} \\ 0 & 0 & 0 & \frac{27}{320} \\ 0 & 0 & 0 & \frac{7}{90} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{323}{1440} & -\frac{11}{120} & \frac{53}{1440} & -\frac{19}{2880} \\ \frac{29}{360} & \frac{31}{90} & \frac{1}{15} & -\frac{1}{360} \\ \frac{27}{320} & \frac{51}{160} & \frac{9}{40} & -\frac{3}{320} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{bmatrix}$$

Kindly refer to [3] for the analysis of the basic properties of the block method.

2.3 Implementation of the Method

In order to implement the method, we propose a prediction equation of the form

$$Y_m^{(0)} = e y_n + h^{\lambda} \sum_{\lambda=1}^{2} \frac{\partial^{\lambda}}{\partial x^{\lambda}} f(x, y) \quad (x_0, y_0) \qquad (9)$$

Substituting (9) into the general block formula (5) gives

$$Y_m = ey_n + h\left[df\left(y_n\right) + bF\left(Y_m^{(0)}\right)\right] \qquad (10)$$

Writing (10) in a linearized form gives

$$Y_{N+1} = Y_{N+\mu} + h\left[bF(Y_{N+\mu})\right] \qquad (11)$$

Where μ is the grid points, $Y_{N+\mu}$ are the interpolation points and $F(Y_{N+\mu})$ are the collocation points, hence substituting (10) into (11) gives

$$Y_{N+1} = Y_{N+\mu} + h\left[bF(Y_m)\right] \qquad (12)$$

Hence (12) is our corrector.

3.0 Analysis of the Basic Properties of the Corrector

3.1 Order and Error Constant of the Corrector

Let the linear operator $L\{y(x):h\}$ associated with the hybrid linear multistep method be defined as

$$L\{y(x):h\} = Y_m - ey_n - h\left[df(y_n) + bF(Y_m^{(0)})\right] \qquad (13)$$

Expanding (13) in Taylor's series and comparing the coefficient of h gives

$$L\{y(x):h\} = C_0 y(x) + C_1 y^1(x) + \ldots + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + \ldots$$

Definition 3: The linear operator L and associated block formula are said to be of order p if $C_0 = C_1 \ldots C_p = C_{p+1} = 0, C_{p+1} \neq 0. C_{p+1}$ is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+1} h^{p+1} y^{p+1}(x) + 0(h^{p+2})$. For our corrector $C_0 = C_1 = \ldots C_8 \neq 0.$ hence the order of the corrector is 7 with error constant -3.8468×10^{-8}.

3.2 Zero Stability of the Corrector

Definition 4: A block method is said to be zero stable if as $h \to 0$, the roots, $r_j = 1(1)k$ of the first characteristics polynomials $p(x) = 0$, that is $p(r) = \det\left[\sum A^{(0)} R^{k-1} - 1\right] = 0$ satisfying $|R| \leq 1$ must have multiplicity equal to unity. The first characteristics polynomial of (7) is

$$P(r) = r + \frac{27}{17} + \frac{64}{17} r^{\frac{1}{4}} - \frac{108}{17} r^{\frac{1}{2}} \qquad (14)$$

equating (14) to zero and solving for r gives the roots of the first characteristics polynomial 0 and $1.$ Hence our corrector is zero-stable.

3.3 Consistency of the Corrector

The method (12) is consistent since it has order $p = 7 > 1$

3.4 Convergence

The method (12) is convergent by the convergence of Dahlquist theorem given below.

3.4.1 Theorem

The necessary and sufficient conditions that a continuous linear multistep method be convergent are that it must be consistent and zero-stable.

4.0 Numerical Experiments

In this section, our concern is the application of the predictor corrector on some initial value problems with test problems 4.1.1- 4.1.3. The program was coded in MATLAB R2010a Software.

4.1.1 Problem: $y' = -10(y-1)^2,\ y(0) = 2,\ y(x) = 1 + \frac{1}{(1+10x)},\ h = 0.01,\ 0 \le x \le 1.$

[source: 3, 4]

4.2.1 Problem: $y' = xy,\ y(0) = 1,\ y(x) = e^{\frac{x^2}{2}},\ h = 0.1,\ 0 \le x \le 1.$

[source: 3, 15]

4.1.3 Problem:

[source: 3, 13, 16]

Table 4.1: Showing the results for problem 4.1.1

X	Error in [4]	Error in [3]	Error in predictor-corrector
0.01	1.07(-03)	9.527220(-005)	9.856992(-006)
0.02	2.38(-03)	7.520349(-005)	5.161066(-005)
0.03	2.21(-03)	7.990705(-005)	6.693997(-005)
0.04	5.36(-03)	7.803322(-005)	7.055929(-005)
0.05	7.53(-03)	7.346380(-005)	6.897749(-005)
0.06	9.00(-03)	6.798272(-005)	6.519366(-005)
0.07	9.98(-03)	6.241065(-005)	6.062286(-005)
0.08	1.06(-02)	5.711225(-005)	5.143879(-005)
0.09	1.10(-02)	5.223270(-005)	5.053870(-005)
0.10	1.12(-02)	4.781142(-005)	4.726448(-005)

Table 4.2: Showing the results for problem 4.1.2

X	Error in [15]	Error in [3]	Error in predictor corrector
0.10	5.29(-007)	2.606759(-011)	5.495604(-013)
0.20	1.77(-007)	8.431988(-011)	2.750444(-011)
0.30	8.99(-007)	1.850877(-010)	8.826473(-011)
0.40	3.09(-007)	3.479586(-010)	1.947702(-010)
0.50	1.91(-006)	6.051188(-010)	3.692060(-010)
0.60	4.48(-006)	1.006964(-009)	6.480878(-010)
0.70	1.02(-005)	1.630994(-009)	1.089035(-009)
0.80	7.74(-005)	2.595632(-009)	1.781515(-009)
0.90	1.44(-005)	4.081569(-009)	2.863633(-009)
1.00	2.93(-005)	6.364684(-009)	4.548275(-009)

Table 4.3: Showing the results for problem 4.1.3.

h	X	Error in predictor corrector	Error in [3]	Error in [13]	Error in [16]
$\frac{1}{16}$	10	0.199(-03) 1.248(-13)	0.199(-03) 1.279(-13)	0.199(-03) 4.470(-09)	0.199(-03) 1.700(-10)
$\frac{1}{8}$	10	0.199(-03) 9.695(-12)	0.199(-03) 1.018(-11)	0.199(-03) 4.515(-08)	0.199(-03) 3.090(-09)
	20	0.499(-04) 6.214(-13)	0.499(-04) 6.370(-13)	0.499(-04) 2.938(-09)	0.499(-04) 1.950(-10)
$\frac{1}{4}$	10	0.199(-03) 8.956(-10)	0.199(-03) 9.956(-10)	0.199(-03) 8.987(-07)	0.199(-03) 4.937(-08)
	20	0.499(-04) 5.922(-11)	0.499(-04) 6.224(-11)	0.499(-04) 5.732(-08)	0.499(-04) 3.107(0.09)

5.0 Discussion of Result

Problem 4.1.1 was solved by [3, 4] where a three block backward differentiation formula and hybrid block integrator were proposed. Problem 4.1.2 was solved by [3, 15] where a stiff starting block method of order six and hybrid block integrator were proposed. Problem 4.1.3 was solved by [3, 13, 16].The method proposed by [3, 13, 16] are of order five, four and six respectively. Table 4.1, 4.2, 4.3 shows clearly that our method gives better approximation than the existing method.

6.0 Conclusion

We have proposed a new method that combines the properties of predictor corrector method and the block method in this paper. It has been established that increasing the interpolation points with the same block predictor gives better approximation.

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