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The Accuracy and Convergence of Canonical Polynomial Method for Solving First Order Differential Equations

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Abstract

In this paper we seek a canonical polynomial for solving first order differential equation. A canonical polynomial was derived using a simple recursive relation and its associated initial value conditions with respect to step by step procedures of both shifted Legendre and shifted Chebyshev polynomials.These were used to solve first order differential equations with initial value problems. The results obtained from these methods were satisfying convergence properties as the error difference tends to zero when compared with the analytical solution.

Keywords: Canonical Polynomial, Shifted Chebyshev Polynomial and Shifted Legendre Polynomial.

1.0 Introduction

Various techniques of solving ordinary differential equation abound, but largely is either by analytical or numerical method it is noted that not all ordinary differential equations can be solved analytically, when analytical solution not possible, numerical method employed [1]. In this paper more interests may need to be observed on the derivation of canonical polynomial techniques other than the existing ones using a simple recursive relation and its associated initial value conditions with respect to step by step procedures of both shifted Legendre and shifted Chebyshev polynomials.Adequate emphasis would be laid on the derivation of canonical polynomial method.

2.0 Derivation of Canonical Polynomials

Let φ_m(x) define a set of polynomial called the canonical polynomials. We shall consider an ordinary differential equation of the form [2]

$y'(x) = f(x, y)$ with initial condition $y(0) = 1, 0 \le x \le 1$ $\quad (1)$

The polynomials are derived using equation (1) by substituting a finite power expansion of the form

$$y(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + \cdots + b_nx^n + \cdots \qquad (2)$$

Putting an error terms x^m on the right side of the equation and we substitute equation (2) and its first derivative into equation (1) we obtained the resultant coefficients.

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$$\left.\begin{array}{l} b_0 - b_1 = 0 \\ b_1 - 2b_2 = 0 \\ b_2 - 3b_3 = 0 \\ b_3 - 4b_4 = 0 \\ \quad\vdots \end{array}\right\} \qquad (3)$$

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$$y_m = \varphi_m = m!\,(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^m}{m!}) \qquad (5)$$

3.0 Shifted Chebyshev Polynomials

Suppose the interval of integration of $T_n^*(x)$ is generated from [-1, 1] to [a, b]. We require a linear transformation in the interval [a, b] which is given in [3] as $t = \frac{a + b - 2x}{a-b}$
Then, for a special interval [0, 1], t = 2x – 1
Therefore, the shifted Chebyshev polynomial is

$$T_n^*(x) = \text{Cos}\,(n\text{Cos}^{-1}\,(2x - 1)) \text{ when } n = 1, 2, 3, \ldots \qquad (6)$$

It has recursion formula

$$T_{n+1}^* = 2(2x-1)T_n^*(x) - T_{n-1}^*(x)$$

When n = 0, $T_0^*(x) = 1$
When n = 1, $T_1^*(x) = 2x - 1$
When n = 2, $T_2^*(x) = 8x^2 - 8x + 1$
When n = 3, $T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$

Clearly, the leading coefficient of the shifted Chebyshev polynomial is 2^{2n-1} so that equation (6) is written as

$$T_n^*(x) = \sum_m {}^nC_n^m x^m \quad , \quad n = 0,1,2,3, \qquad (7)$$

The coefficients of the shifted Chebyshev polynomial are obtained as follows:
For n = 0
$T_0^*(x) = \sum {}_0^0C_0^0 x^0 = 1 \qquad \rightarrow C_0^0 = 1$
For n = 1
$T_1^*(x) = C_1^1 x + C_1^0 x^0 = 2x - 1 \qquad \rightarrow C_1^1 = 2, C_1^0 = -1$
For n = 2
$T_2^*(x) = C_2^2 x^2 + C_2^1 x + C_1^0 x^0 = 8x^2 - 8x + 1 \rightarrow C_2^2 = 8, C_2^1 = -8, C_2^0 = 1$
And so on. The under-listed are shifted Chebyshev polynomials:

1
$2x - 1$
$8x^2 - 8x + 1$
$32x^2 - 48x^2 + 18x - 1$
$128x^4 - 256x^3 + 160x^2 - 32x + 1$
$512x^5 - 1280x^4 + 112x^3 - 400x^2 + 50x - 1$
$2043x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1$
$8192x^7 - 28672x^6 + 3942x^5 - 26880x^4 + 9403x^3 - 1568x^2 + 98x - 1$
$32768x^8 - 131072x^7 + 212992x^6 - 180224x^5 + 81480x^4 - 21504x^3 + 2688x^2 - 128x + 1$

4.0 Shifted Legendre Polynomials

The shifted Legendre polynomials are defined as: p_n (x) = p_n (2x – 1) where the "Shifting function". x→ 2x – 1, is chosen such that it bijectively maps the interval [0,1] to the interval [-1,1] implying that the polynomial $p_n(x)$ are orthogonal on [0,1] which is given in [4]

$$\int_0^1 P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$$

An explicit expression for the shifted Legendre polynomials is given by:

$$p_n(x) = (-1)^n \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-x)^k,\ n = 0,1,2,3... \quad (8)$$

The analogue of Rodrigue's formula for the shifted Legendre polynomial is

$$p^*_n(x) = (n!)^{-1}\frac{d^n}{dx^n}\left[\left(x^2 - x\right)^n\right] \quad (9)$$

Thus,

$P_0(x) = 1$

$P_1(x) = \frac{1}{1}\frac{d}{dx}[x^2 - x] = 2x - 1$

$P_2(x) = \frac{1}{2}\frac{d^2}{dx^2}[x^2 - x]^2 = 6x^2 - 6x + 1$

$P_3(x) = \frac{1}{3!}\frac{d^3}{dx^3}[x^2 - x]^3 = 20x^3 - 30x^2 + 12x - 1$

$P_4(x) = \frac{1}{4!}\frac{d^4}{dx^4}[x^2 - x]^4 = 1680x^4 - 3360x^3 2160x^2 - 480x + 24$

$P_5(x) = \frac{1}{5!}\frac{d^5}{dx^5}[x^2 - x]^5 = 252x^5 - 630x^4 + 280x^3 - 210x^2 + 30x - 1$

$P_6(x) = \frac{1}{6!}\frac{d^6}{dx^6}[x^2 - x]^6 = 942x^6 - 2772x^5 + 3150x^4 - 1680x^3 + 420x^2 - 42x + 1$

$P_7(x) = \frac{1}{7!}\frac{d^7}{dx^7}[x^2 - x]^7$

$= 3432x^7 - 12012x^6 + 16632x^5 - 11550x^4 + 4200x^3 - 756x^2 + 56x - 1$

$P_8(x) = \frac{1}{8!}\frac{d^8}{dx^8}[x^2 - x]^8$

$= 12870x^8 - 51480x^7 + 84084x^6 - 72072x^5 + 34650x^4 - 9249x^3 + 1260x^2 - 72x + 1$

5.0 The Numerical Solution for ODEs Using Canonical Polynomials

To obtain the general solution of equation (1) using canonical polynomial and the shifted Legendre polynomial we re-write equation (1) in the form [5]

$$Dy = 0 \quad (10)$$

If we put an error term $\tau p_n^*(x)$ on the right of equation (8), we have

$$Dy = \tau p_n^*(x) \quad (11)$$

where $\tau p_n^*(x)$ is a shifted Legendre Polynomial and is given by

$$\tau P_n^*(x) = \sum_{m=0}^{n} L_n^m x^m \quad (12)$$

If by substituting equation (10) into equation (9), we have

$$Dy = \tau \sum_{m=0}^{n} L_n^m x^m \quad (13)$$

L_n^m is the coefficient of shifted Legendre polynomial.
We now integrate equation (11) with respect to x to get

$$y = \tau \sum_{m=0}^{n} L_n^m \varphi_m(x) \quad (14)$$

We determine τ by using the initial condition as above

$$\tau = \frac{1}{\sum_{m=0}^{n} L_n^m m!} \quad (15)$$

Thus

$$y = \frac{\sum_{m=0}^{n} L_n^m \varphi_m(x)}{\sum_{m=0}^{n} L_n^m m!} \quad (16)$$

Equation (14) is the explicit solution of equation (9) using the shifted Legendre polynomial written as a linear superposition of canonical polynomials associated with the given differential operator. The coefficient of the shifted Legendre polynomials are just a weighting factor of these polynomials.
Then the explicit solution becomes

$$y_n = \sum_{m=0}^{n} L_n^m \left(\tau_1\phi_n(x) + \tau_2\phi_{n+1}(x)\right) \quad (17)$$

The equation (17) is the general solution of (10).
We shall now obtain the solution for (1) using the shifted Chebyshev polynomial to approximation our solution. The associated canonical polynomial are the same with equation (5) and the general solution in this case is given as

$$y = \tau \sum_{m=0}^{n} C_n^m \phi_m(x) \quad (18)$$

Where C_n^m are the coefficients of the shifted Chebyshev polynomial and we now obtain our solution y_n as

$$y_n(x) = \frac{\sum_{m=0}^{n} C_n^m m! A_m}{\sum_{m=0}^{n} C_n^m m!} \quad (19)$$

6.0 Test Problem

We shall now obtain the solution for (1) using the shifted Chebyshev polynomial.

$$y - y' = 0 \text{with initial condition} y(0) = 1 \quad (20)$$

The associated canonical polynomials are given in equation (5). The general solution is provided in equation (17). Thus, the solutions using shifted Chebyshev polynomials are

$y_0 = 1$

$y_1 = 1 + 2x$

$y_2 = \frac{9 + 8x + 8x^2}{9}$

$y_3 = \frac{113 + 114x + 48x^2 + 32x^3}{113}$

$y_4 = \frac{1825 + 1824x + 928x^2 + 256x^3 + 128x^4}{1825}$

$y_5 = \frac{36689 + 36690x + 18320x^2 + 6240x^3 + 1280x^4 + 512x^5}{36689}$

$y_6 = \frac{879673 + 879672x + 439872x^2 + 395176x^3 + 37482x^4 + 6114x^5 + 2043x^6}{87673}$

$y_7 = \frac{20525139 + 20531412x + 10262521x^2 + 3421363x^3 + 852990x^4 + 175974x^5 + 28672x^6 + 8192x^7}{20525139}$

Similarly, the solution to equation (1) using the shifted Legendre polynomial is provided in equation (16). Thus, the solutions using shifted Legendre polynomial are

$P_0(x) = 1$

$P_1(x) = 2x + 1$

$P_2(x) = \frac{6x^2 + 6x + 7}{7}$

$P_3(x) = \frac{20x^3 + 30x^2 + 72x + 71}{71}$

$P_4(x) = \frac{70x^4 + 140x^3 + 510x^2 + 1000x + 100}{100}$

$P_5(x) = \frac{252x^5 + 630x^4 + 2800x^3 + 8190x^2 + 16410x + 16409}{16409}$

$P_6(x) = \frac{942x^6 + 2880x^5 + 17550x^4 + 68520x^3 + 205980x^2 + 411919}{411919}$

The above values of y's are the approximate solutions to equation (17)
The analytical solution of equation (1) is

$$y_n(x) = y(x) = e^x \quad (21)$$

The approximated solutions are compared with the exact solution of equation (18) for $y(0)=1, 0 \le x \le 1$, the results obtained are as follows intable 1,2,3 and 4 for each y_i = 1,2,3, …

Table 1: Shifted Chebyshev and Shifted Legendre polynomial for y_1

	Shifted Legendre	**Shifted Chebyshev**	**Analytical**		
X	$Y_1^L = 1+2x$	$Y_I^C = 2x+1$	$y = e^x$	$E_L = \lvert Y_I^L - y\rvert$	$E_C = \lvert Y_I^C - y\rvert$
0	1	1	1	0	0
0.1	1.200000	1.200000	1.105171	$9.4829\text{x}10^{-2}$	$9.4829\text{x}10^{-2}$
0.2	1.400000	1.400000	1.221403	$1.78597\text{x}10^{-1}$	$1.78597\text{x}10^{-1}$
0.3	1.600000	1.600000	1.349859	$2.50141\text{x}10^{-1}$	$2.50141\text{x}10^{-1}$
0.4	1.80000	1.800000	1.491825	$3.08175\text{x}10^{-1}$	$3.08175\text{x}10^{-1}$
0.5	2.000000	2.000000	1.648721	$3.51279\text{x}10^{-1}$	$3.51279\text{x}10^{-1}$
0.6	2.200000	2.200000	1.822119	$3.7881\text{x}10^{-1}$	$3.7881\text{x}10^{-1}$
0.7	2.400000	2.400000	2.013753	$3.8624\ \text{x}10^{-1}$	$3.8624\ \text{x}10^{-1}$
0.8	2.600000	2.600000	2.225541	$3.74459\text{x}10^{-1}$	$3.74459\text{x}10^{-1}$
0.9	2.800000	2.800000	2.459602	$3.40397\text{x}10^{-1}$	$34o397\text{x}10^{-1}$
1.0	3.000000	3.000000	2.718282	$2.81718\text{x}10^{-1}$	$2.81718\text{x}10^{-1}$

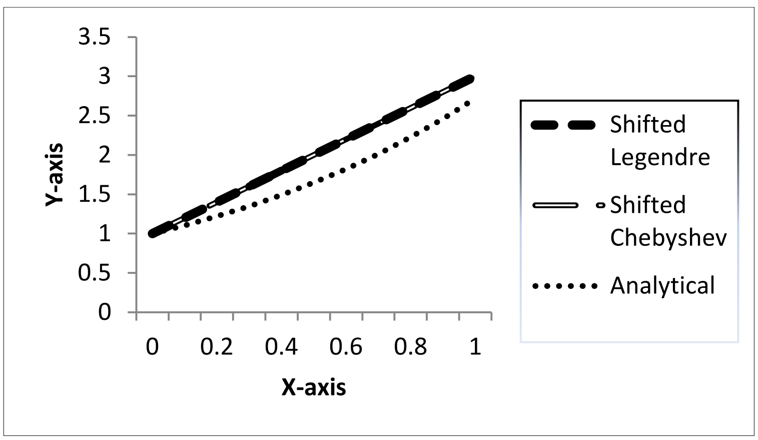

Fig 1: Shifted Chebyshev and Shifted Legendre polynomial for $y_{1,}$

Remark: Figure 1, Shows that the shifted chebyshev and the shifted legendre polynomials are good approximation when compared to the analytical solution.

Table 2: Shifted Chebyshev and Shifted Legendre polynomial for y_2

	Shifted Legendre	**Shifted Chebyshev**	**Analytical**		
X	$Y_2^L = {}^1\!/_7\,(6x^2+6x+7)$	$Y_2^C = {}^1\!/_9\,(9+8x+8x^2)$	$y = e^x$	$E_L = \lvert Y_2^L - y\rvert$	$E_C = \lvert Y_2^C - y\rvert$
0	1	1	1	0	0
0.1	1.094286	1.09778	1.105171	$1.885\text{x}10^{-2}$	$7.393\text{x}10^{-3}$
0.2	1.205714	1.213333	1.221403	$1.5689\text{x}10^{-2}$	$8.07\text{x}10^{-3}$
0.3	1.334286	1.346667	1.349859	$1.5573\text{x}10^{-2}$	$3.192\text{x}10^{-3}$
0.4	1.48000	1.497778	1.491825	$1.7778\text{x}10^{-2}$	$5.951\text{x}10^{-3}$
0.5	1.642857	1.666667	1.648721	$2.381\text{x}10^{-2}$	$1.7946\text{x}10^{-2}$
0.6	1.822857	1.853333	1.822119	$3.0476\text{x}10^{-2}$	$3.1214\text{x}10^{-2}$
0.7	2.020000	2.057778	2.013753	$6.247\ \text{x}10^{-3}$	$4.4025\text{x}10^{-2}$
0.8	2.234286	2.280000	2.225541	$8.745\ \text{x}10^{-3}$	$5,4459\text{x}10^{-2}$
0.9	2A65714	2.250000	2.459603	$6.111\text{x}10^{-3}$	$6.0397\text{x}10^{-2}$
1.0	27 14286	2.777778	2.718282	$3.996\ \text{x}10^{-3}$	$5.9496\text{x}10^{-2}$

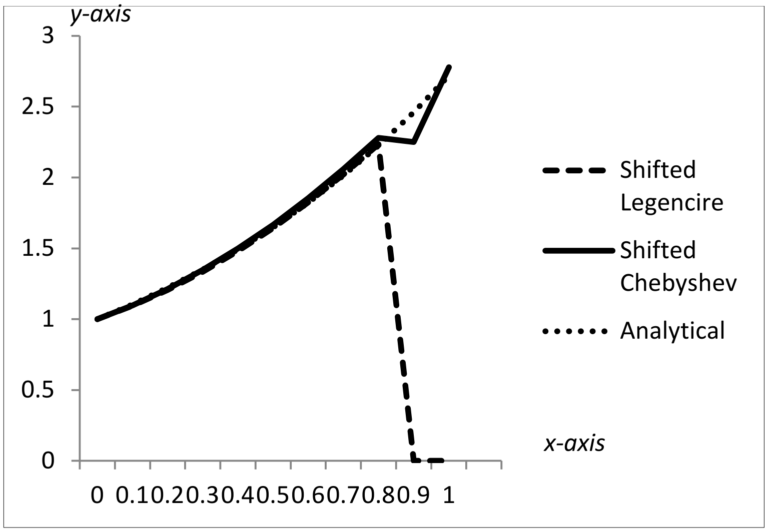

Fig 2: Shifted Chebyshev and Shifted Legendre polynomial for y_2

Remarks: Fig 2, Shows that the shifted Chebyshev is more accurate than the shifted Legendre

Table 3: Shifted Chebyshev and Shifted Legendre polynomial for y_3

	Shifted Legendre	**Shifted Chebyshev**	**Analytical**		
X	$Y_3^L = \frac{1}{7}\left[20x^3+30x^2+72x+71\right]$	$Y_3^C = {}^1/_{113}[113+114x+48x^2+32x^3]$	$y = e^x$	$E_L = \lvert Y_{24}^L - y\rvert$	$E_C = \lvert Y_4^C - y\rvert$
0	1	1	1	0	0
0.1	1.105915	1,105416	1.105171	$7.44\text{x}10^{-4}$	$2.45\text{x}10^{-4}$
0.2	1.221972	1.221027	1.221403	$5.69\text{x}10^{-4}$	$9.45\text{x}10^{-4}$
0.3	1.349859	1.348531	1.349859	0.0000	$1.380\text{x}10^{-3}$
0.4	1.491268	1.489628	1.491825	$5.57\text{x}10^{-4}$	$5.57\text{x}10^{-4}$
0.5	1.647887	1.646018	1.648721	$8.34\ \text{x}10^{-4}$	$2.703\text{x}10^{-3}$
0.6	1.821408	1.819398	1.822119	$7.11\ \text{x}10^{-4}$	$2.729\text{x}10^{-3}$
0.7	2.013521	2.011469	2.013753	$2.32\ \text{x}10^{-4}$	$2.284\text{x}10^{-3}$
0.8	2.225915	2.223929	2.225541	$3.74\ \text{x}10^{-4}$	$1.612\text{x}10^{-3}$
0.9	2.460282	2.458478	2.459602	$6.8\text{x}10^{-4}$	$1.124\text{x}10^{-3}$
1.0	2.718310	2.716814	2.718282	$2.8\text{x}10^{-5}$	$1.468\text{x}10^{-3}$

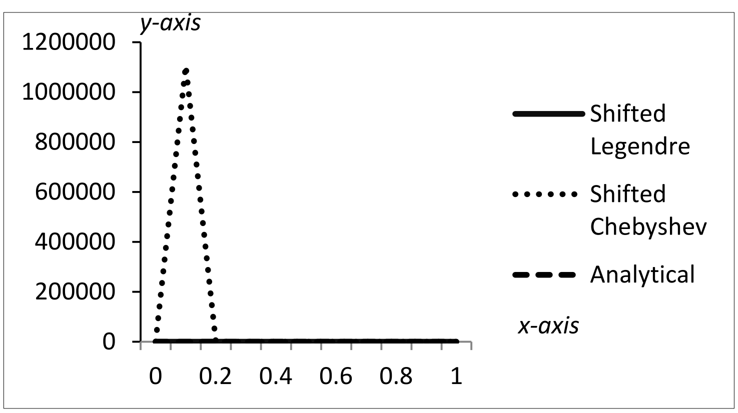

Fig 3: Shifted Chebyshev and Shifted Legendre polynomial for y_2 .

Remarks: The graph shows that the shifted Legendre polynomial is accurate than the Shifted Chebyshev.

Table 4: Shifted Chebyshev and Shifted Legendre polynomial for y_4

	Shifted Legendre	**Shifted Chebyshev**	**Analytical**		
X	$Y_4^L = {}^1/_{100}\left[70x^4 + 140x^3 + 510x^2 + 1000x + 1001\right]$	$Y_4^C = {}^1/_{1825}$ $[18259 + 1824x + 928x^2 + 3256x^3 + 128x^4]$	$y = e^x$	$E_L = \lvert Y_4^L - y \rvert$	$E_C = \lvert Y_4^C - y \rvert$
0	1	1	1	0	0
0.1	1.105142	1.105177	1.105171	$2.9\text{x}10^{-5}$	$6\text{x}10^{-6}$
0.2	1.221411	1.221464	1.221403	$8.0\text{x}10^{-5}$	$6.1\text{x}10^{-5}$
0.3	1.349897	1.349955	1.349859	$3.8\text{x}10^{-5}$	$9.6\text{x}10^{-5}$
0.4	1.491860	1.491913	1.491825	$3.4\text{x}10^{-5}$	$8.8\text{x}10^{-5}$
0.5	1.648726	1.648787	1.648721	$5\text{x}105^{-6}$	$4.6\text{x}10^{-5}$
0.6	1.822090	1.822118	1.822119	$2.9\text{x}10^{-5}$	$1\text{x}10^{-6}$
0.7	2.013713	2.013732	2.013753	$4\text{x}10^{-5}$	$2.\text{lxlO}^{-5}$
0.8	2.225526	2.225546	2.225541	$1.5\text{x}10^{-5}$	$5\text{x}10^{-6}$
0.9	2.459627	2.459663	2.459602	$2.5\text{x}10^{-5}$	$6.1 \text{ x } 10^{-5}$
1.0	2.718282	2.718356	2.718282	0	$7.4\text{x}10^{-5}$

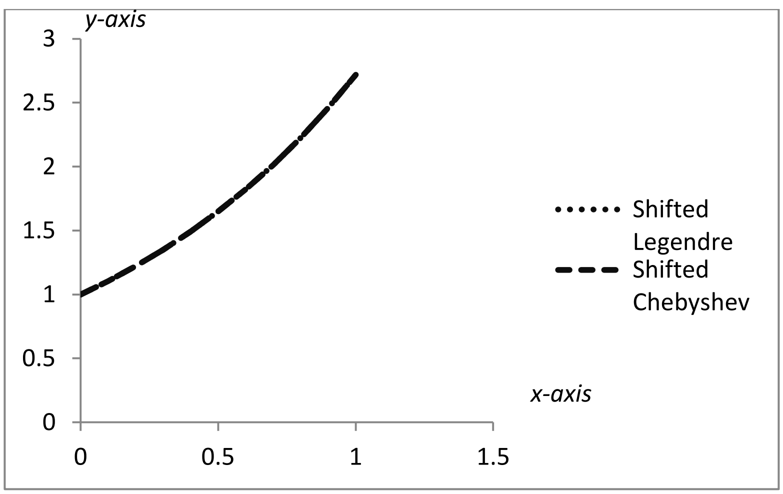

Fig 4: Shifted Chebyshev and Shifted Legendre polynomial for y_4,
Remarks: Fig 4 , Shows that the shifted chebyshev and the shifted Legendre polynomials are accurate When compared to the analytical solution

7.0 Conclusion

The canonical polynomial Q_m (x) provides an excellent approximation to an ordinary differential equation of the initial value type. It fails when the general term Q_m(x) cannot be obtained. We also see that the shifted Legendre polynomial and the shifted Chebyshev polynomial are very useful in approximating our solution, y_n(X). They provide solutions which are very close to the analytical solution. Shifted Chebyshev and Shifted Legendre polynomial for y_4 give the same results as the analytical solution. Hence, they provide the best solution to initial value problem in ODEs.

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