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Approximate solution of nth –order Initial Value Problems by an Iterative Decomposition Method

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Abstract

Approximate solution of a class of nth-order Initial value Problems (IVP'S) are considered, applying an Iterative Decomposition method. The method, which gives solution as rapidly convergent infinite series of easily computable terms requires no linearization or discretization. Some examples are presented to establish the accuracy and efficiency of the proposed method.

Keywords: Initial Value Problems, Decomposition, Error

1.0 Introduction

Initial Value Problems of Higher Order Ordinary differential Equations occur in several mathematical models of problems in Physics, Engineering and Technology. In general, higher order initial value Problems are not easily solved [1,2]. It is possible to solve an nth-order IVP by reducing it to a system of first order IVP'S and applying any known method available for such problems [1, 2, 3, 4].

However, it is quite desirable to provide direct numerical methods for solving nth-order IVP's.

In this work, we consider a class of nth-order IVP's of the form

$$x^{(n)}(t) = f\left(t, x(t), x^1(t), x^{11}(t), \ldots, x^{(n-1)}(t)\right) \quad (1)$$

Subject to the initial conditions

$$x(t_0) = x_0,\; x^1(t_0) = x^1{}_0, \ldots\; x^{(n-1)}(t_0) = x^{(n-1)}{}_0, \qquad (2)$$

Some recent direct numerical methods have been applied to solve $(1)-(2)$. For example, a variable step Runge-Kutta-Nystrom method was applied in [2]. Furthermore, in [3] the semi-numeric multistage modified Adomian Decomposition Method was applied. In the same way, Homotopy methods have been applied to the same problems. For example, in [1] the applicability of the Homotopy Perturbation Method (HPM) for the solution of the nth-order IVP's is demonstrated. In [5] the solutions were obtained by the Homotopy Analysis Method (HAM).

These methods give favourable results, showing reliable convergence to exact solutions or the closed forms of the exact solutions in most cases. See Ref [6] for example.

The proposed method has been found to be accurate and efficient for some classes of Ordinary Differential equations. For example, in [7] the method was applied to solve the one-dimensional Biharmonic equation. The solution of Variational Problems was considered in [8] and in [9] Delay Differential Equations were considered.

In this paper, we apply the Iterative Decomposition Method to approximate the nth-order Initial Value Problems, directly. The major motivation for this work is the need for a solution technique, which can be applied with relative ease, requiring minimal mathematical rigour or details. The proposed algorithm presents the solutions in the form of rapidly convergent infinite series of easily computable terms. The organization of the rest of this paper is as follows: In section 2, we present an analysis of the Iterative Decomposition Method for nth-order Initial Value problems. To present a clear overview of the method, we apply the algorithm developed in section 2 to some examples with known analytical solutions in section 3. A conclusion is presented in section 4.

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2.0 The Iterative Decomposition Method

Consider the nth-order IVP

$$y^{(n)}(x) = f\left(x, y(x), y^1(x), y^{11}(x), \ldots y^{(n-1)}(x)\right) \qquad (3)$$

$$y(0) = \alpha_{0,}\ y^1(0) = \alpha_{1,}\ \ldots,\ \ y^{(n-1)}(0) = \alpha_{n-1} \qquad (4)$$

Where α_0, α_1, …, α_{n-1} are given constants, and f is a continuous, real, linear or non linear function. Equation (3) can be written in the form

$$Ly(x) = f(x, y(x), y^1(x), \ldots, y^{(n-1)}(x)) \qquad (5)$$

Where the differential operator L is given as

$$L(.) = \frac{d^n}{dx^n}(.) \qquad (6)$$

The operator L is assumed invertible and the inverse operator $L^{-1}(.)$ is thus an n-fold integral operator defined by

$$L^{-1}(.) = \int_0^x \int_0^x \ldots \int_0^x (.) dx dx \ldots dx \qquad (7)$$

Operating the inverse operator (7) on (5), it follows that

$$y(x) = \sum_{n=0}^{n-1} \frac{\alpha_n}{n!} x^n + L^{-1}\left\{f\left(x, y(x), y^1(x), \ldots, y^{(n+1)}(x)\right)\right\} \qquad (8)$$

The Iterative Decomposition Method assumes that the unknown function $y(x)$ can be expressed in terms of an infinite series of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \qquad (9)$$

so that the components $y_n(x)$ can be determined iteratively. To convey the idea and for the sake of completeness of the iteration technique [7], we can see that (8) is of the form

$$Ny = L^{-1}(y) + k \qquad (10)$$

Where k is a constant and $N(y)$ is the nonlinear term as

$$\sum_{i=0}^{\infty} y_i = L^{-1}(y_0) + \sum_{j=0}^{\infty} \left\{L^{-1}\left(\sum_{j=0}^{n} y_j\right) - L^{-1}\left(\sum_{j=0}^{n-1} y_j\right)\right\} \qquad (11)$$

From (9) and (11), (10) is equivalent to

$$\sum_{i=0}^{\infty} y_i = k + L^{-1}(y_0) + \sum_{j=0}^{\infty} \left\{L^{-1}\left(\sum_{j=0}^{n} y_j\right) - L^{-1}\left(\sum_{j=0}^{n-1} y_j\right)\right\} \qquad (12)$$

are thus,

$$
\begin{aligned}
y_0(x) &= k \\
y_1(x) &= L^{-1}(y_0) \\
y_2(x) &= L^{-1}(y_0 + y_1) - L^{-1}(y_0) \\
&\;\;\vdots
\end{aligned}
\tag{13}
$$

$$y_{n+1} = L^{-1}(y_0 + y_1 + \ldots + y_n) - L^{-1}(y_0 + y_1 + \ldots + y_{n-1})$$, n=1, 2, . . . Thus,

$$y = k + \sum_{n=1}^{\infty} y_n \tag{14}$$

The zeroth component $y_0(x)$ is defined through all terms that arise from the initial conditions (2).

3.0 Numerical Examples

To illustrate the efficiency and accuracy of the IDM, we shall consider some examples.

Example 3.1

Consider the nonlinear second order IVP [6]

$$y^{11}(t) + \left(y^{1}(t)^{2}\right) = 0 \tag{15}$$

Subject to the initial conditions

$$y(0) = 1, y^{1}(0) = 2$$

The exact solution is

$$y(t) = 1 + \ln(1 + 2t) \tag{16}$$

By the IDM,

$$y(t) = 1 + 2t - L^{-1}\left\{\left(y^{1}\right)^{2}\right\} \tag{17}$$

Where $L = \frac{d^2}{dt^2}$

Taking $y_0(t) = 1 + 2t$,

We have

$$
\begin{aligned}
y_1(t) &= -2t^2 \\
y_2(t) &= \frac{8t^3}{3} - \frac{4t^4}{3} \\
y_3(t) &= \frac{-8t^4}{3} + \frac{64t^5}{15} - \frac{32t^6}{9} + \frac{128t^7}{63} - \frac{32t^8}{63}
\end{aligned}
\tag{18}
$$

.

Then, $y(t)$ can be approximated as

$$y(t) = 1 + 2t - 2t^2 + \frac{8t^3}{3} - 4t^4 + \frac{64t^5}{15} - \frac{32t^6}{9} + \frac{128t^7}{63} - \frac{32t^8}{63} + \cdots \tag{19}$$

Table 1: Table of Values for Example 3.1

T	Exact Solution	Approximation solution IDM	Error
0.0	1.000000000	1.000000000	0
0.1	1.182321557	1.182305976	1.558E-5
0.2	1.336472237	1.336395817	7.642E-5
0.3	1.470003629	1.469999417	4.412E-5
0.4	1.587786665	1.581055472	3.712E-6
0.5	1.693147181	1.693140181	7.00E-6
0.6	1.78845736	1.78840336	5.40E-5
0.7	1.875468737	1.87549471	2.597E-5
0.8	1.955511445	1.95549447	1.696E-5
0.9	2.029619417	2.029598161	2.126E-5
1.0	2.098612289	2.098158217	2.54E-4

Example 3.2
Consider the linear fourth-order IVP [2]

$$y^{(iv)}(t) = -5y^{11} - 4y \qquad (20)$$

Subject to the initial conditions

$$y(0) = 1, y^{1}(0) = 0, y^{11}(0), y^{111}(0) = 1 \qquad (21)$$

The exact solution is

$$y(t) = \frac{4}{3}\cos t + \frac{1}{3}\sin t - \frac{1}{3}\cos 2t - \frac{1}{6}\sin 2t \qquad (22)$$

In operator form (20) can be written as

$$L(y) = -5y^{11} - 4y \qquad (23)$$

Where $L = \frac{d^4}{dt^4}$

Applying the appropriate inverse operator L^{-1} to both sides,

$$y(t) = 1 + \frac{t^3}{6} - L^{-1}\{5y^{11} + 4y\} \qquad (24)$$

Taking $y_0(t) = 1 + \frac{t^3}{6},$ we have

$$y_1(t) = -\frac{t^4}{6} - \frac{t^5}{12} - \frac{t^7}{1260}$$

$$y_2(t) = \frac{t^4}{3} + \frac{t^5}{24} + \frac{t^6}{36} + \frac{5t^7}{504} + \frac{t^9}{18144} + \frac{t^{11}}{2494800} \qquad (25)$$

.
.
.
And so on.
Then, $y(t)$ can be approximated as

$$y(t)=1+\frac{t^3}{6}-\frac{t^4}{6}-\frac{t^5}{24}+\frac{t^6}{36}+\frac{5t^7}{504}+\frac{t^9}{18144}+\frac{t^{11}}{2494800}+\ldots \quad (26)$$

Table2: Table of Values for Example 3.2

t	Exact Solution	Approximate Solution by IDM	Error
0.0	1.000000000	1.000000000	0
0.1	1.000149612	1.000149612	0
0.2	1.001055159	1.001055238	7.90E-8
0.3	1.003069771	1.003071171	1.40E-6
0.4	1.006092521	1.00610338	1.086E-5
0.5	1.009572663	1.009626224	3.356E-5
0.6	1.012535879	1.01273427	1.984E-4
0.7	1.013631476	1.014234342	6.029E-4
0.8	1.01119855	1.012783016	1.584E-3
0.9	1.003348354	1.007074861	3.729E-3
1.0	1.9880594438	0.9960868607	8.027E-3

Example 3.3

Consider the nonlinear fourth-order IVP

$$y^{(iv)}(t)=yy^1+\left(y^1\right)^2 \quad (27)$$

Subject to the initial conditions

$$y(0)=0, y^1(0)=1, y^{11}(0)=1, y^{111}(0)=1 \quad (28)$$

The exact solution is

$$y(t)=e^t-1 \quad (29)$$

By IDM,

$$y(t)=t+\frac{t^2}{2!}+\frac{t^3}{3!}+L^{-1}\left\{\left(y^1\right)^2-yy^{11}\right\}$$

Taking $y_0(t)=t+\frac{t^2}{2}+\frac{t^3}{6}$,

We have

$$y_1(t)=\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{2560}+\frac{t^8}{20160}$$

$$y_2(t)=\frac{-t^7}{5040}-\frac{t^8}{40320}+\frac{t^9}{181440}-\frac{t^{10}}{403200}-\frac{t^{11}}{497920}-\frac{t^{12}}{17107200}-\frac{t^{13}}{207567360}+\frac{t^{14}}{1320883200}+\frac{t^{15}}{19813248000}+\frac{t^{16}}{39626496000}-\frac{t^{17}}{7972194462}+\frac{t^{18}}{370987008000} \quad (30)$$

and so on.

Thus, $y(t)$ can be approximated as

$$y(t)=t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{6}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{t^7}{5040}+\frac{t^8}{40320}+\frac{t^9}{181440}-\frac{t^{10}}{40320}-\frac{t^{11}}{497920}-\frac{t^{12}}{1710200}-\frac{t^{13}}{207567360}+\frac{t^{14}}{1320883200}+\frac{t^{15}}{19813248000}+\frac{t^{16}}{39626496000}-\frac{t^{17}}{7972194462}+\frac{t^{18}}{370987008000}.$$

Table3: Table of Values for Example 3.3

t	Exact Solution	Approximate Solution by IDM	Error
0.0	0.000000000	0.000000000	0
0.1	0.10517089181	0.1051709181	0
0.2	0.2214027582	0.2214027582	0
0.3	0.3498588076	0.3498588076	0
0.4	0.4918246976	0.491824698	3.884E-10
0.5	0.6487212707	0.6487212724	1.683E-9
0.6	0.8221188004	0.822118804	3.584E-9
0.7	1.013752707	1.013752701	6.406E-9
0.8	1.225540926	1.225540828	9.836E-8
0.9	1.459603111	1.459603321	1.01E-8
1.0	1.718281828	1.718281801	2.70E-8

4.0 Discussion and Conclusion

In this work, an Iterative Decomposition Method has been applied to solve nth-Order Initial Value Problems. The method is quite easy to handle, and guarantees convergence of the approximate solution to the exact solution, even for very few terms of the approximating series. The numerical examples considered have illustrated the accuracy and efficiency of the method.

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