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Analytic Solutions of a Special Class of Nonlinear Partial Differential Equations

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Abstract

An analytic method of finding a closed form solutions of a special class of nonlinear partial differential equations in two independent variables is presented. A special condition is imposed on the coefficients which makes it possible to obtain intermediate integrals through which a closed form solution is obtained by further integration. The viability of the method is demonstrated with typical examples.

Keywords: Analytic solution, closed form solution, Nonlinear PDE, Monge method, developable

1.0 Introduction

For many classes of nonlinear partial differential equations, explicit or closed form solutions are very difficult to come by. Any justifiable method is considered viable. Unfortunately most real life models would always lead to solving nonlinear partial differential equations. Many scholars who are confronted with this problem have resorted to using some linear approximations, numerical methods or some other approximate methods. Most of the known analytic methods can only be applicable to a very small range of nonlinear partial differential equations.

Rubel [1] presented a method that seeks the solutions through what he described as 'quasi solution'. The method involves considering an equation of a higher other which is relatively easy to solve and its solution must contain the solution of the given partial differential equation after a finite number of differentiations. Rubel's method is yet to be exploited or advanced probably because of its clumsy nature. In [2] Schetcher used the Legendre transformation for functions of more than one variable to find closed form solutions of some classes of nonlinear partial differential equations. However this method is not applicable to developable surfaces. Erumaka [3] expounded the method of simple waves outlined by John [4]. Although a number of model problems of applied physics can be solved using the later method but it can only be applied to nonlinear partial differential equations reducible to the form:

$$f(x, y, u, u_x)\, u_{xx} = u_{yy}$$

Unfortunately not all nonlinear partial differential equations can be reduced to the above form. Johnson and Smoller [5] presented a method that can only be applied to the hyperbolic equations or equations reducible to its form.

A good number of other analytic methods with their attendant limitations in application can be seen elsewhere in literature see for example [6 - 13].

As a lee way, a good number of authors have resorted to using numerical methods with all its unavoidable constraints. In consequence numerical schemes abound in literature. Some examples can be seen in [14 - 19].

In the first part of the presentation in [17] we show how the method popularly referred to as the 'Monge method' can be used to find the closed form solutions of a larger class of nonlinear partial differential equations via analytic procedure. In that paper we limited the method to a degenerate case when the equation is uniformly nonlinear.

In the present paper, considered as the concluding part, we consider the method of [20] as developed in [21] for a general form of the nonlinear partial differential equation. As in the first part of this work, we have included in the concluding part of the present paper solutions of some nonlinear partial deferential equations as a demonstration of the viability of this method.

2.0 Methodology

Given the general partial differential equation in two independent variables x and y (say) of the form

$$F(x, y, u, p, q, r, s, t) = 0 \qquad (2.1)$$

where

$$p = u_x,\ q = u_y,\ r = u_{xx},\ s = u_{xy},\ t = u_{yy}$$

and

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$$u = u(x, y) \tag{2.2}$$

is a continuous function of the independent variables x and y.
This method is based on establishing some first integrals of the form

$$\zeta = f(\xi) \tag{2.3}$$

where
$\zeta = \zeta\,(x, y, u, p, q)$
$\xi = \xi\,(x, y, u, p, q)$
and f is an arbitrary function of ξ such that (2.1) can be derived from (2.3). By differentiating (2.3) with respect to x and y respectively and eliminating f′ between the two derivatives, see [21], we obtain the ratio.

$$\frac{\zeta_x + p\zeta_u + r\zeta_p + s\zeta_q}{\zeta_y + q\zeta_u + t\zeta_q + s\zeta_p} = \frac{\xi_x + p\xi_u + r\xi_p + s\xi_q}{\xi_y + q\xi_u + t\xi_q + s\xi_p} \tag{2.4}$$

A careful simplification of equation (2.4) gives the relation

$$rR + sS + tT + (rt - s^2)U = V \tag{2.5}$$

where

$$R = \frac{\partial(\xi,\zeta)}{\partial(p,\mathrm{u})} + q\,\frac{\partial(\xi,\zeta)}{\partial(p,\mathrm{u})}$$

$$S = \frac{\partial(\xi,\zeta)}{\partial(q,\mathrm{y})} + q\frac{\partial(\xi,\zeta)}{\partial(p,\mathrm{u})} + \frac{\partial(\xi,\zeta)}{\partial(p,\mathrm{x})} + p\frac{\partial(\xi,\zeta)}{\partial(p,\mathrm{u})}$$

$$T = \frac{\partial(\xi,\zeta)}{\partial(q,\mathrm{u})} + p\frac{\partial(\xi,\zeta)}{\partial(q,\mathrm{u})}$$

$$U = \frac{\partial(\xi,\zeta)}{\partial(p,\mathrm{q})}$$

$$U = \frac{\partial(\xi,\zeta)}{\partial(x,\mathrm{y})} + q\frac{\partial(\xi,\zeta)}{\partial(u,\mathrm{x})} + p\frac{\partial(\xi,\zeta)}{\partial(u,\mathrm{y})}$$

A successive differentiation of equation (2.2) and some simplifications give

$$r = \frac{dp - sdy}{\mathrm{dx}} \tag{2.6}$$

and

$$t = \frac{dp - sdx}{\mathrm{dy}} \tag{2.7}$$

If we substitute equations (2.6) and (2.7) into (2.5) we obtain the relation

$$\begin{aligned} &R(dp - sdy)dy + s\,Sdxdy + T(dq\text{-}sdx)\,dx \\ &+ U[(dp - sdy)(dq - sdx) - s^2dxdy] = V\,dx\,dy \end{aligned} \tag{2.8}$$

It is easy to see that equation (2.8) is valid only if

$$Rdpdy + T\,dqdx + Udqdp = 0 \tag{2.9}$$

and

$$R(dy)^2 - S\,dxdy + T(dx)^2 + Udpdx + Udqdy = 0 \tag{2.10}$$

We can express equations (2.9) and (2.10) as one in the form

$$\begin{aligned} &R(dy)^2 - (S + \lambda V)\,dxdy + T(dx)^2 + Udpdx + U\,dqdy \\ &+ \lambda Rdpdy + \lambda Tdqdx + \lambda Udqdp = 0 \end{aligned} \tag{2.11}$$

where λ is a parametric multiplier to be determined.
If we choose λ such that

$$\lambda^2\,(RT + UV) + \lambda US + U^2 = 0 \tag{2.12}$$

we see that equation (2.11) can be expressed as product of two factors.

$$(U\,dy + \lambda T\,dx + \lambda Udp)(\,\lambda Rdy + Udx + \lambda Udq) = 0 \tag{2.13}$$

Let λ_1 and λ_2 be the roots of equation (2.12), we are then led to the following two possible systems of equations

$$\begin{cases} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{cases} \quad (2.14)$$

and

$$\begin{cases} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{cases} \quad (2.15)$$

From each pair (2.14) and (2.15) we derive two intermediate integrals of the form

$$\xi(x, y, u, p, q) = c_1$$
$$\zeta(x, y, u, p, q) = c_2$$

and hence a first integral of the form (2.2) from each pair, which in turn can be solved to determine p and q as factors of x, y and u. By substituting the result into the differential of u namely

$$du = pdx + qdy \quad (2.16)$$

and integrating, we arrive at the complete integral containing arbrary functions, which is the general solution of (2.1).

3.0 Illustrative Examples

In this section, we demonstrate the application of the method outlined in section 2 in finding closed form solutions to some nonlinear partial differential equations arising from real life models.

Example 1. Solve the equation

$$3r + s + t + rt - s^2 = -9 \quad (3.1)$$

If we compare equation (3.1) with (2.5) we find that

$$R = 3, \quad S = 1, \quad T = 1, \quad U = 1 \quad \text{and } V = -9$$

When these values are used in condition (2.12) we have the two values of λ as

$\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{3}$

When the values of λ are used in equation (2.14) we obtain the system of partial differential equation

$$\begin{cases} dy + \frac{1}{2}dx + \frac{1}{2}dp = 0 \\ \\ dx - dy - \frac{1}{3}dq = 0 \end{cases} \quad (3.2)$$

Solving (3.2) simultaneously, we obtain the first intermediate integral as

$$2y + x + p = f(q + 3y - 3x) \quad (3.3)$$

Also using the values of λ in equation (2.15) we obtain the system

$$\begin{cases} dy - \frac{1}{3}dx - \frac{1}{3}dp = 0 \\ \\ dx - \frac{3}{2}dy + \frac{1}{2}dq = 0 \end{cases} \quad (3.4)$$

with solution as

$$p + x - 3y = f(q + 3y + 2x) \quad (3.5)$$

We may combine (3.3) and (3.5) to have

$$y = \frac{1}{5}\left[f(\alpha) - f(\beta)\right] \quad (3.6)$$

where

$$\alpha = q + 3y - 3x \quad (3.7)$$

and

$$\beta = q + 3y + 2x. \quad (3.8)$$

From equation (3.6) we obtain

$$dy = \frac{1}{5}\left[f'(\alpha)\, d\alpha - f'(\beta)\, d\beta \right] \qquad (3.9)$$

while equation (3.5) now reads

$$p = f(\alpha) - x - 2y \qquad (3.10)$$

Now using equation (2.16) we have

$$du = [f(\alpha) - x - 2y]dx + (\beta - 2x - 3y)dy \qquad (3.11)$$

Using equations (3.5) – (3.10) in equation (3.11) we obtain

$$du = -(xdx + 2\, d(xy) + 3ydy) - \frac{1}{5} f(\alpha)\, dx$$

$$+\frac{1}{5} d\left[\beta\, f(\alpha)\right] - \beta\, f'(\beta)\, d\beta \qquad (3.12)$$

By integrating equation (3.12) we obtain the required solution as

$$2u = -x^2 - 4xy - 3y^2 - \phi(\alpha) + \beta\left[\phi'(\alpha) - \psi'(\beta)\right] + \psi(\beta)$$

where

$$\phi(\alpha) = \int f(\alpha)dx \quad \text{and} \quad \psi(\beta) = \int f(\beta)d\beta .$$

Example 2. Solve the equation

$$u(1 + q^2)r - 2pqus + u(1 + p^2)t + (rt-5^2)u^2 + 1 + p^2 + q^2 = 0 \qquad (3.13)$$

Compared with (2.5) we obtain

$$R = u(1 + q^2), \quad S = -2pqu, \quad T = u(1 + p^2), \; U = u^2$$

and $V = -(1 + p^2 + q^2)$

substituting in equation (2.12) we have

$$p^2q^2\lambda^2 - 2pqu\lambda + u^2 = 0 \qquad (3.14)$$

Hence $\lambda = \dfrac{u}{pq}$ is the only root. Since we have only one root, only one pair of equations (2.14) and (2.15) is used to determine the intermediate integral. Therefore, using system (2.14) we obtain the system of partial differential equations

$$\begin{cases} pqdy + (1 + p^2)\, dq + udp = 0 \\ \\ (1 + q^2)\, dy + pqdx + udq = 0 \end{cases} \qquad (3.15)$$

Integrating (3.15) and using (2.16) we obtain the general solution of (3.13) as

$$u^2 = x^2 + y^2 - 2c_1x - 2c_2y + k. \qquad (3.16)$$

Example 3. Solve the equation

$$r + 4s + t(1 + r) - s^2 = 2 \qquad (3.17)$$

Compared with (2.5) we have

$$R = 1, \quad S = 4, \quad T = 1, \quad U = 1 \quad \text{and} \quad V = 2$$

and the condition (2.12) for λ gives

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = \frac{1}{3}$$

Substituting in turn in the pairs (2.14) and (2.15) and integrating the system of resulting system of partial differential equations we obtain the two intermediate integrals

$$y - x - p = h(y - 3x + q) \qquad (3.18)$$

$$y - x + q = g(3y - x - p) \qquad (3.19)$$

where

$$y - x - p = c_1 \qquad (3.20)$$

and

$$y - x + q = c_2 \qquad (3.21)$$

are particular integrals. We see here that it appears difficult to solve for p and q explicitly from (3.20) and (3.21). Hence we may combine one integral with a particular integral of the other. To achieve a solution we rewrite (3.21) as

$$y - x + q = g(2y + c_1)$$

Finally we use equation (2.16) to obtain the complete integral as

$$u = c_1x + c_2y + x^2 + y^2 - 2xy + G(2y + c_1) + c_3 \qquad (3.22)$$

as the closed form solution of equation (3.17)

4.0 Summary and Conclusion

In our illustrative examples, we have concentrated in equations of the form (2.5). This of course is proper since the equation will have to be reduced to that form before the method could be applied. However in section 2, we have also outlined how the second order partial differential equation, (2.1), of a general form can be reduced to the form (2.5) by some appropriate transformations.

When the roots of equation (2.12) are coincidental, we need only one of the pairs (2.14) and (2.15) to generate the intermediate integral which in this case may required some caution in other to get the complete integral. The idea is demonstrated in example 2. In general, the method is more general in nature and application. Its viability can always be tested. It therefore commands a good advantage over other known methods.

References

[1] Rubel L .A. Closed form solution of some partial differential equations via quasi-solutions 1. Illinois J. of Math., 33 (2),1991.

[2] Schechter M. Modern methods in partial Differential Equations. McGraw-Hill, London, 1977.

[3] Erumaka E.N. On simple wave solution for a class of nonlinear waveforms. J. Math. Sci., 18(2), 2007.

[4] JohnF .Partial Differential Equation. Springer-Verlag, New York, 1982.

[5] Johnson J.H. and Smoller J.A. Global solutions for certain system of quasi-linear hyperbolic equations. J.Math. Mech., 17:561-576, 1969-467, 1969.

[6] Bragg L. R. and John W. D. Related problems in partial differential equations and their applications. J. Soc. Indus. Appl. Math.,16:459-467, 1968.

[7] Carrol R. C. Abstract Methods in Partial Differential Equations. Harper and Row , London, 1969.

[8] Chester C. R Techniques in partial Differential Equation McGraw-Hill Kogakusha Ltd.,1971.

[9] Courant R. and Hilbert D.Methods of Mathematical Physics, volume II.Inter Science, N.Y. 1966

[10] Garabedian P.R. Partial Differential Equation 2nd ed. Chelsea, New York, 1986.

[11] John F.and Chester C. Elements of Partial Differential Equations Springer-Verlag, New York,1991.

[12] Lucas J., Barbosa M., and Colares A.G. Minimal surface in R3,Lecture notes in Mathematics, No. 1195. Springer-Verlag, New York, 1986.

[13] Osserman R. A Survey of Minimal Surfaces. Van Nostrund, 1969.

[14] Ames W. F. Numerical Methods for Partial Differential Equations . Academic Press, New York, N.Y. 1977.

[15] Briggs W.L. and Henson V.E. A multigrid tutorial SIAM, 1973.

[16] Dresner L. Similarity Solution of nonlinear partial Differental Equations. Pitman, London, 1983.

[17] Forsyth G.E. and Wasow W. R. Finite Difference Methods of partial Differential Equation. John Wiley, New York, 1960.

[18] The Meis and Marcowitz U. Numerical Solution of Partial Differential Equations. Springer-Verlag, New York, 1981.

[19] Zacharow V.E. and Shabat A.B.A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem. Functional Anal. Apps, 8:226-235, 1976.

[20] Onugha E.E and E.N. Erumaka. Closed form solution of some nonlinear partial differential equations. J. Nig. Asso. Math. Phy., 25:47-50,2013.

[21] Erumaka E.N. On the monge method for nonlinear partial differential equation. J. Math. Sci., 22 (2), 2010