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Newton's Equations of Motion for Particles of Non Zero Rest Masses in The Gravitational Field of a Static Homogeneous Prolate Spheroidal Distribution of Mass

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Abstract

The theoretical study of gravitation is well known in the fields of massive bodies of perfect spherical geometry, based on the assumption that the Earth is a perfect sphere. But it is well known that the only reason for these restrictions is mathematical convenience and simplicity. The real fact of nature is that all rotating planets, stars and galaxies in the universe are spheroidal and the motions of test particles in their gravitational fields require the use of spheroidal coordinates. In this paper, we derive Newton's equations of motion for test particles in Newton's gravitational field of a static homogeneous prolate spheroidal distribution of mass to pave the way for the corresponding extension of the well-known mechanics in spherical gravitational fields to spheroidal gravitational fields.

Keywords: non zero rest masses, gravitational field, prolate spheroidal.

1.0 Introduction

Newton's theory of universal gravitation (TUG) was restricted almost exclusively to the fields of massive bodies of perfect spherical geometry. For example, in the solar system the motion of bodies (such as planets, comets and asteroids) are treated under the assumption that the sun is a perfect sphere. Also in the General Relativity Theory (GRT), the motion of bodies (such as planets) and particles (such as photons) are treated under the assumption that the sun is a perfect sphere (Schwarzschild's space-time) [1]. The real fact of Nature is that all rotating stars and planets, and galaxies in the universe are spheroidal. It is obvious that their spheroidal geometry will have significant effects in the motions of all particles in their gravitational fields. These effects will exist in both Newtonian mechanics and in Einstein's theory. In this paper, we hereby pave way for the solution of the equations of motion of test particles in the gravitational fields of prolate spheroidal bodies.

2:0 Mathematical Analysis

Consider a homogeneous prolate spheroidal body of rest mass M_0. Then the prolate spheroidal coordinates (η, ξ, ϕ) are defined in terms of the Cartesian coordinates (x, y, z) by [2,3,4]

$$x = a\,(1-\eta^2)^{\frac{1}{2}}(\xi^2-1)^{\frac{1}{2}}\cos\phi \tag{1}$$

$$y = a\,(1-\eta^2)^{\frac{1}{2}}(\xi^2-1)^{\frac{1}{2}}\sin\phi \tag{2}$$

$$z = a\eta\xi \tag{3}$$

where a is a constant and

$$0 < \xi < \infty, -1 \leq \eta \leq 1,\ 0 \leq \phi \leq 2\pi \tag{4}$$

Also in the spheroidal coordinates, the surface of the spheroidal as given by

$$\xi = \xi_0 \tag{5}$$

where ξ_0 is a constant.

New if the body is homogeneous its density, ρ, is given by

$$\rho(\underline{r}) = \rho_0\,;\ \xi \leq \xi_0 \tag{6}$$

and

$$\rho(\underline{r}) = 0\,;\ \xi > \xi_0 \tag{7}$$

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where ρ_0 is the constant density of rest mass. As it is well known, Newton's gravitational field equation for the gravitational scalar potential f_g due to a distribution of mass density ρ_0 is given by

$$\nabla^2 f_g = 4\pi G\rho_0 \tag{8}$$

where G is the universal gravitational constant. It follows from the explicit expression for the Laplacian operator in prolate spheroidal coordinates that the interior and exterior and gravitational scalar potentials, f^- and f^+ respectively satisfy the equations:

$$4\pi G\rho_0 = \left\{\frac{1}{a^2(\xi^2-\eta^2)}\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta} + \frac{1}{a^2(\xi^2-\eta^2)}\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial}{\partial\xi}\right]\right.\right.$$
$$\left.\left.+\frac{1}{a^2}\frac{\partial}{\partial\phi}\left[\frac{1}{(1-\eta^2)(\xi^2-1)}\frac{\partial}{\partial\phi}\right]\right]\right\} f^-(\eta,\xi,\phi) \tag{9}$$

and

$$0 = \left\{\frac{1}{a^2(\xi^2-\eta^2)}\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta} + \frac{1}{a^2(\xi^2-\eta^2)}\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial}{\partial\xi}\right]\right.\right.$$
$$\left.\left.+\frac{1}{a^2}\frac{\partial}{\partial\phi}\left[\frac{1}{(1-\eta^2)(\xi^2-1)}\frac{\partial}{\partial\phi}\right]\right]\right\} f^+(\eta,\xi,\phi) \tag{10}$$

By symmetry of a homogeneous prolate spheroidal distribution of the mass about the polar axis, the gravitational potential will be independent of coordinate ϕ.
It follows that (9) and (10) are reduced to:

$$4\pi G\rho_o a^2(\xi^2-\eta^2) = \frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}f^-(\eta,\xi) + \frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}f^-(\eta,\xi) \tag{11}$$

and

$$0 = \frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}f^+(\eta,\xi) + \frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}f^+(\eta,\xi) \tag{12}$$

General complementary solution of (11) is given as:

$$f_c^-(\eta,\xi) = \sum_{l=0}^{\infty}\left[A_l^- P_l(\xi) + B_l^- Q_{(\xi)}\right]\left[C_l^- P_l(\eta) + D_l^- Q_{(\eta)}\right] \tag{13}$$

We seek particular solution of (11) as:

$$f_\rho^-(\eta,\xi) = B\,(\xi^2+\eta^2) \tag{14}$$

By using (13) in (10), we have

$$B = \frac{2}{3}a^2\pi G\rho_0 \tag{15}$$

Hence general solution of (11) and (12) are given as:

$$f^-(\eta,\xi) = \sum_{l=0}^{\infty}\left[A_l^- P_l(\xi) + B_l^- Q_{(\xi)}\right]\left[C_l^- P_l(\eta) + D_l^- Q_l(\eta)\right] + B\,(\xi^2+\eta^2) \tag{16}$$

and

$$f^+(\eta,\xi) = \sum_{l=0}^{\infty}\left[A_l^+ P_l(\xi) + B_l^+ Q_{(\xi)}\right]\left[C_l^+ P_l(\eta) + D_l^+ Q_l(\eta)\right] \tag{17}$$

where $A_l^-, B_l^-, C_l^-, D_l^-$ *and* $A_l^+, B_l^+, C_l^+, D_l^+$ are arbitrary constants, and P_l and Q_l are the two linearly independent Legendre functions of order $l = 0, 1, 2, \ldots.$
Now since the interior and exterior regions both contain the coordinate η=0, which is a singularity of Q_l we choose:

$$D_l^- \equiv D_l^+ \equiv 0; l = 0, 1, 2, \ldots. \tag{18}$$

in the general solution of (16) and (17). Also, since ξ=0 is a singularity of Q_l we choose:

$$B_l^- \equiv 0; l = 0, 1, 2, \ldots \tag{19}$$

Also, since P_l is not defined for ξ ⟶∞ in the exterior region, we choose:

$$A_l^+ \equiv 0; l = 0, 1, 2, \ldots \tag{20}$$

It follows that (16) and (17) becomes:

$$f^-(\eta,\xi) = \sum_{l=0}^{\infty}\left[A_l^- P_l(\xi) C_l^- P(\eta)\right] + B\,(\xi^2+\eta^2)$$
$$= \sum_{l=0}^{\infty} A_l P_l(\xi) P_l(\eta) + B\,(\xi^2+\eta^2) \tag{21}$$

and

$$f^{+}(\eta,\xi)=\sum_{l=0}^{\infty}[B_l^{+}Q_l(\xi)C_l^{+}P_l(\eta)]=\sum_{l=0}^{\infty}B_lQ(\xi)P_l(\eta) \qquad (22)$$

where A_l and B_l are arbitrary constants. Consequently by the conditions of the continuity of the potentials and their normal derivatives at the $\xi=\xi_0$, boundary of the spheroid, it follows that:

$$A_0=\frac{B\left[2\,Q_0(\xi_0)\xi_0-\left(\xi_0^2+\frac{1}{3}\right)Q_0^{|}(\xi_0)\right]}{Q_0^{|}(\xi_0)P_0(\xi_0)-Q_0(\xi_0)P_0^{|}(\xi_0)} \qquad (23)$$

$$A_2=\frac{\frac{2}{3}BQ_0^{|}(\xi_0)}{Q_2(\xi_0)P_2^{|}(\xi_0)-P_{2(\xi_0)}\,Q_2^{|}(\xi_0)} \qquad (24)$$

$$A_1=B_1=0 \qquad (25)$$

$$B_0=\frac{B\left[2\,P_0(\xi_0)\xi_0-P_0^{|}(\xi_0)(\xi_0^2+\frac{1}{3})\right]}{Q_0^{|}(\xi_0)P_0(\xi_0)-Q_{0(\xi)}P_0^{|}(\xi_0)} \qquad (26)$$

$$B_2=\frac{\frac{2}{3}BP_2^{|}(\xi_0)}{Q_2(\xi_0)P_2^{|}(\xi_0)-P_{2(\xi_0)}Q_2^{|}(\xi_0)} \qquad (27)$$

and

$$A_l=B_l\,;l=1,3,4,5\,.... \qquad (28)$$

Consequently the final solutions are:

$$f^{-}(\eta,\xi)=A_0P_0(\eta)P_0(\xi)+A_2P_2(\eta)P_2(\xi)+B\,(\xi^2+\eta^2) \qquad (29)$$

and

$$f^{+}(\eta,\xi)=B_0P_0(\eta)Q_0(\xi)+B_2P_2(\eta)Q_2(\xi) \qquad (30)$$

These are the Newtonian interior and exterior gravitational scalar potentials of the prolate spheroidal in terms of its constant rest mass density ρ_0 and surface coordinate ξ_0 and parameter a.

Newton's equations of motion, in prolate spheroidal coordinates are defined as [5]:

$$\underline{a^{-}}(\eta,\xi,\phi)=-\nabla f^{-}(\eta,\xi,\phi) \qquad (31)$$

and

$$\underline{a^{+}}(\eta,\xi,\phi)=-\nabla f^{-}(\eta,\xi,\phi) \qquad (32)$$

where $\underline{a}$ is the instantaneous acceleration in terms of prolate speroidal coordinates given as:

$$\underline{a}=a_\eta\hat{\eta}+a_\xi\hat{\xi}+a_\phi\hat{\phi} \qquad (33)$$

$$\underline{a}_\eta=a\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{\frac{1}{2}}\left\{\ddot{\eta}+\frac{2\xi}{\xi^2-\eta^2}\dot{\eta}\dot{\xi}+\frac{\eta\,(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\dot{\eta}^2+\frac{\eta\,(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\dot{\xi}^2+\frac{\eta\,(1-\eta^2)(\xi^2-1)}{(\xi^2-\eta^2)}\dot{\phi}^2\right\}\hat{\xi} \qquad (34)$$

$$\underline{a}_\xi=a\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{\frac{1}{2}}\left\{\ddot{\xi}-\frac{2\eta}{\xi^2-\eta^2}\dot{\eta}\dot{\xi}-\frac{\xi\,(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\dot{\eta}^2-\frac{\xi\,(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\dot{\xi}^2-\frac{(\xi^2-1)(1-\eta^2)}{(\xi^2-\eta^2)}\dot{\phi}^2\right\}\hat{\xi} \qquad (35)$$

and

$$\underline{a}_\phi=a[(1-\eta^2)(\xi^2-1)]^{\frac{1}{2}}\left\{\ddot{\phi}-\frac{2\eta}{(1-\eta^2)}\dot{\eta}\dot{\phi}+\frac{2\xi}{(\xi^2-1)}\dot{\xi}\dot{\phi}\right\}\hat{\phi} \qquad (36)$$

Also, the del or nebla $(\underline{\nabla})$ is expressed in terms of prolate spheroidal coordinates as[5]:

$$\nabla(\eta,\xi,\phi)=\frac{\hat{\eta}(1-\eta^2)^{\frac{1}{2}}}{a(\xi^2-\eta^2)^{\frac{1}{2}}}\frac{\partial}{\partial\eta}+\frac{\hat{\xi}(\xi^2-1)^{\frac{1}{2}}}{a(\xi^2-\eta^2)^{\frac{1}{2}}}\frac{\partial}{\partial\xi}+\frac{\hat{\phi}}{a[(1-\eta^2)(\xi^2-1)]^{\frac{1}{2}}}\frac{\partial}{\partial\phi} \qquad (37)$$

Then

$\underline{a}_\eta^- = -(\underline{\nabla}\mathrm{f}^-)_\eta$

$$\ddot{\eta} + \frac{2\xi}{\xi^2-\eta^2}\dot{\eta}\dot{\xi} + \frac{\eta\,(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\dot{\eta}^2 + \frac{\eta\,(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\dot{\xi}^2 + \frac{\eta\,(1-\eta^2)(\xi^2-1)}{\xi^2-\eta^2}\phi^2 + \eta(3A_2P_2(\xi)+2B) = 0 \qquad (38)$$

and

$\underline{a}_\xi^- = -(\underline{\nabla}\mathrm{f}^-)_\xi$

$$\ddot{\xi} - \frac{2\eta}{\xi^2-\eta^2}\dot{\eta}\dot{\xi} - \frac{\xi\,(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\dot{\eta}^2 - \frac{\xi\,(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\dot{\xi}^2 - \frac{\xi\,(\xi^2-1)(1-\eta^2)}{\xi^2-\eta^2}\dot{\phi}^2 + A_0Q_0^{|}(\xi) + \frac{1}{2}A_2(3\eta^2-1)Q_2^{|}(\xi) + 2B\xi = 0 \qquad (39)$$

$\underline{a}_\phi^- = -(\underline{\nabla}\mathrm{f}^-)_\phi$

$$\ddot{\phi} - \frac{2\eta}{1-\eta^2}\dot{\eta}\,\dot{\phi} + \frac{2\xi}{\xi^2-1}\dot{\xi}\,\dot{\phi} = 0 \qquad (40)$$

$\underline{a}_\eta^+ = -(\underline{\nabla}\mathrm{f}^+)_\eta$

$$\ddot{\eta} + \frac{2\xi}{\xi^2-\eta^2}\dot{\eta}\,\dot{\xi} + \frac{\eta\,(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\dot{\eta}^2 + \frac{\xi\,(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\dot{\xi}^2 + \frac{\eta\,(1-\eta^2)(\xi^2-1)}{\xi^2-\eta^2}\dot{\phi}^2 + B_2Q_2(\xi)\eta = 0 \qquad (41)$$

and

$\underline{a}_\xi^+ = -(\underline{\nabla}\mathrm{f}^+)_\xi$

$$\ddot{\xi} - \frac{2\eta}{\xi^2-\eta^2}\dot{\eta}\dot{\xi} - \frac{\xi\,(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\dot{\eta}^2 - \frac{\xi\,(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\dot{\xi}^2 - \frac{\xi\,(\xi^2-1)(1-\eta^2)}{\xi^2-\eta^2}\dot{\phi}^2 + B_0Q_0^{|}(\xi) + \frac{1}{2}B_2Q_2^{|}(\xi)(3\eta^2-1) = 0 \qquad (42)$$

$\underline{a}_\phi^+ = -(\underline{\nabla}\mathrm{f}^+)_\phi$

$$\ddot{\phi} - \frac{2\eta}{1-\eta^2}\dot{\eta}\,\dot{\phi} + \frac{2\xi}{\xi^2-1}\dot{\xi}\,\dot{\phi} = 0 \qquad (43)$$

We integrate (43) exactly to yield:

$$\dot{\phi} = \frac{L}{(1-\eta^2)(\xi^2-1)} \qquad (44)$$

where L is a constant. It follows that (41) and (42) becomes:

$$\ddot{\eta} + \frac{2\xi}{\xi^2-\eta^2}\dot{\eta}\,\dot{\xi} + \frac{\eta\,(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\dot{\eta}^2 + \frac{\eta\,(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\dot{\xi}^2 + \frac{\eta L^2}{(1-\eta^2)(\xi^2-1)(\xi^2-\eta^2)} + 3B_2Q_2(\xi)\eta = 0 \qquad (45)$$

and

$$\ddot{\xi} - \frac{2\eta}{\xi^2-\eta^2}\dot{\eta}\,\dot{\xi} - \frac{\xi\,(\xi^2-1)}{(1-\eta^2)(\xi^2-\eta^2)}\dot{\eta}^2 - \frac{\xi\,(1-\eta^2)}{(\xi^2-1)(\xi^2-\eta^2)}\dot{\xi}^2 - \frac{\xi L^2}{(\xi^2-\eta^2)(1-\eta^2)(\xi^2-1)} + B_0Q_0^{|}(\xi) + \frac{1}{2}B_2Q_2^{|}(\xi)(3\eta^2-1) = 0 \quad (46)$$

This is the completion of the equations of motion in prolate spheroidal coordinate system.

3:0 Results and Discussion

In this paper we derived the Newton's equations of motion for the interior and exterior scalar gravitational potential in prolate spheroidal coordinate as (45) and (46) respectively.

These equations (45) and (46) extend Newton's theory of classical mechanics from the well-known pure spherical bodies to those of spheroidal bodies, and hence spheroidal effects. Consequently, we have pave way for the theoretical solution of these equations of motions for non-zero rest masses in the gravitational fields of spheroidal bodies, such as the planets and comets and asteroids in the solar system, and satellites in earth orbits.

4.0 Conclusion

Finally, the work in this paper is an excellent demonstration of an application in gravitation theory for orthogonal curvilinear coordinate system other than the usual Cartesian, cylindrical and spherical coordinates..

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