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Identifying out of Control Signals in a Bivariate Control Chart via the Bootstrap Approach

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Abstract

In manufacturing and service industries, most multivariate statistical quality control charts are usually used to determine whether a process is performing as intended or if there are some unnatural causes of variation upon an overall statistics. Once a multivariate control chart signals, identifying which or combination of the quality characteristics responsible for the signal has been a difficult task. To address this problem,a bootstrap approach in setting up control limits is presented, and the decomposition method is presented to identify which or combination of variables responsible for the signals in a bivariate case.

Keywords: bootstrap approach, bivariate control charts, decomposition, signals, identification methods

1.0 Introduction

Generally, univariate control charts involves the computations of one variable. For instance, ifunivariate control chart gives an out of control signal, one can easily detect what the problem is and give a solution since it is related to a single variable. But any decision based on this type of charts when two or more variables are involved is erroneous, hence the multivariate control charts.Most multivariate statistical quality control charts are usually used in manufacturing and service industries to determine whether a process is performing as intended or if there are some unnatural causes of variation upon an overall statistics. Once a multivariate control chart detects out of control signals, the difficulty encountered is to determine whether one or two or a combination of variables is responsible for the abnormal signal. Identifying which or combination of the quality characteristics responsible for the signal becomes necessary in taking appropriate actions that will improve the quality of products. Multivariate control charts are a powerful tool for identifying an out of control process [1]. It has the advantage of being able to monitor multiple quality characteristics simultaneously for both changes in the mean vector and correlation structure while maintaining a specified probability of type 1 error (α).

Some scholars under the parametric technology have been trying to diagnose the abnormal process in multivariate control charts [2 - 4].One of the important multivariate statistical process control tools that have been widely used to monitor multivariate processes is the Hotelling'sT^2 control charts [5 - 10]. Recently, Li et al [11] proposed a framework for diagnosing the out of control signals in multivariate process using optimized support vector machines.

The multivariate control charts assumed that monitoring statistics follow a certain probability distribution such as the multivariate normality assumption. When the distributional assumption is violated (the usual case in practice),a control limits based on these distribution may be inaccurate thereby increasing the rate of false alarms (type 1 error (α)), [12, 13]. Multivariate control charts are at a disadvantage because it is difficult to identify which subset of the quality characteristics responsible for a signal since any single characteristic or combination of characteristics could have experienced a shift in mean value, variance, or correlation,[14-16].

To address these limitations of control charts while retaining their desirable features,a bootstrap approach to set up a control limits is presented for a bivariate case, while the decompositionmethod is used to identify which or combination of variables responsible for the out of control signal. The bootstrap control limits are calculated based on the percentile of statistics derived from bootstrap sample. The bootstrap control limits is easy to implement because it requires neither specification of the parameters nor a procedure for numerical integration. The absence of these requirements makes the bootstrap control chart easier to use.The remaining parts of this paper shall be distributed as: 2.Procedures for Setting Bootstrap Control Limits, 3. Identification of out of Control by Decomposition, 4. Application to Numerical Example, and 5.Concluding Remarks.

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2.0 Procedures for Setting BootstrapControl Limits

Besides the parametric multivariate statistical process control (MSPC) technology, few scholars have been trying to diagnose the abnormal process in multivariate process via the bootstrap based multivariate control charts. A bootstrap control chart that can monitor both dependent and independent observations was proposed in [17] while a discussion of the performance of techniques for constructing bootstrap control charts was done in [18]. A bootstrap control chart based on the Birnbaum-Saunders distribution was proposed in [19] while median control charts whose control limits were determined by estimating the variance of the sample median via bootstrap technique was proposed in [20]. A multivariate T^2 control chart that can monitor a process when the distribution of observed data is not normal or unknown was proposed in [13] while the concept of bootstrapping technique to address the issue of uncorrelated variables as well as the distributional assumptions of minimax control charts was introduced in [21].

In parametric control charting methodology an assumption often used to determine statistical properties is that the data are normally distributed. Assuming the observed process data followed a multivariate normal distribution, the control limit of a T^2 control chart is proportional to the percentile of F-distribution [22]. However, the bootstrap technique is one of the most widely used resampling methods to determine statistical estimators when the population distribution is unknown [13, 23, 24]. What follows is the procedure to obtaining bootstrap control limits:

1. Compute the statistics with n observations from the in control dataset, i.e.
$$T^2 = n(\bar{x} - \bar{\bar{x}})^t S^{-1}(\bar{x} - \bar{\bar{x}}) \quad (1)$$
2. Let $T_1^{2(i)}, T_2^{2(i)}, \dots, T_n^{2(i)}$ be a set of nT^2 values from ith bootstrap sample $(i = 1,2, \dots, B)$ randomly drawn from the initial T^2 statistics with replacement. In general B is the large number $(e.g., B > 1000)$
3. In each B bootstrap samples, determine the $100(1-\alpha)^{th}$ percentile value given a users specified valued α with a range between 0 and 1.
4. Determine the control limit by taking an average of $B100(1-\alpha)^{th}$ percentile values $\bar{T}^2 100(1-\alpha)$.
5. Use the established control limit to monitor a new observation. That is, if the monitoring statistic of a new observation exceeds $\bar{T}^2 100(1-\alpha)$, we declare that specific observation as out of control.

The computational time required has been perceived as one of the disadvantages of the bootstrap technique, but is no longer a difficult issue because of the computing power currently available.

The flow chart in Fig.1 shows an overview of the bootstrap procedure to calculate control limits:

3.0 Identification of Out of Control by Decomposition

Famous literatures in the approach based on the decomposition of the T^2 statistic for the purpose of identifying assignable quality characteristic with a multivariate control chart signal can be seen in [5, 25]. The procedure of [5] decomposes the T^2statistic into two ways for a bivariate data:

$$T^2 = T_1^2 + T_{21}^2 = T_2^2 + T_{12}^2 \quad (2)$$

where
$$T_i^2 = \frac{n(x_i - \bar{x}_i)^2}{\bar{s}_{ii}} \quad (3)$$
$$T_{j.i}^2 = \frac{n(x_j - \bar{x}_{j.i})^2}{s_{j.i}} \quad (4)$$

and$\bar{x}_{j.i}$ and $s_{j.i}$ are the estimators of the conditional mean and variance of $j$for a given value of variable i. The procedure of [25] is to decompose the T^2 statistic into components that reflect the contribution of each individual variable. Thus showing that

$$d_i = T^2 - T_{(i)}^2 \quad (5)$$

is an indicator of relative contribution of the ith variable to the overall statistic, where $T_{(i)}^2$ is the value of the T^2 statistic for all quality characteristics except the ith one. Concretely,

$$T_{(1)}^2 = \frac{n(x_2 - \bar{x}_2)^2}{s_{22}} \quad (6)$$
$$T_{(2)}^2 = \frac{n(x_1 - \bar{x}_1)^2}{s_{11}} \quad (7)$$

and the distribution of d_i is given as

$$d_i \sim \frac{m+1}{m} F_{1,\mathrm{m(n-1)}} \quad (8)$$

where$F_{1,\mathrm{m(n-1)}}$ implies the F distribution with degree of freedoms 1 and $m(n-1)$.

4.0 Application to Numerical Example

The data used in this study comes from a production process with two observable quality characteristics, x_1 and x_2, as shown below. The data are sample means of each quality characteristics, based on samples of size $n = 25$. Assume that the mean values of the quality characteristics and the covariance matrix were computed from 50 preliminarysamples.

$$\bar{x} = \begin{pmatrix} 55 \\ 30 \end{pmatrix}, \qquad s = \begin{pmatrix} 200 & 130 \\ 130 & 120 \end{pmatrix}.$$

The purpose of this study is to set up a bootstrap control limits for monitoring and detection of abnormal behavior. The decomposition is a diagnostic tool to identify which or combination of variables responsible for the out of control when a bivariate control charts signal. The values of the T^2 statistic computed for each sample from (1) are summarized inTable 2.

Using the Bootstrap resampling of T^2 statistic in Table 2, 3000 new samples each of size 15 are drawn with replacement. From the given bootstrap procedures, the bootstrap control limit (CL_B) is computed such that the false alarm rate is fixed to 0.01. The result obtained is $CL_B = 38.63$, and the new Hotelling's T^2 control chart used to monitor future observations is shown in Figure 2.

4.1 Out-of-Control Signal's Interpretation by the Decomposition Method

If the computed statistic is smaller than $CL_B = 38.63$, then the process is still in control, if not, the process is out of control. When the process is out of control, the process will be stopped, and the responsible quality characteristic for this shift should be identified. Then associated assignable causes are detected and eliminated. For testing purposes, additional data are taken from the production process. Statistic T^2 is computed and plotted in control chart as shown in Table 3 and Figure 3 respectively.

In Figure 3, we observed that process is out of control when samples 16, 17, 19 and 20 were taken. To determine which quality characteristic is responsible for the two cases,d_i is computed and given in Table 4.

From Table 4, $d_2$is 37.0159 for sample 16, and by comparing it to the other decomposition value d_1, we notice that the more contributor quality characteristic to the process detected when sample 16 was taken, is the quality characteristic (QC) x_2. In the same way, d_1 is 40.8568 for sample 17, and that the more contributor quality characteristic to the process detected when sample 17 was taken, is the QC x_1, when compared to d_2. However, d_2 is 24.1004 for sample 19, is the only quality characteristic responsible for the out-of-control process, while d_1 for sample 19 is in-control state. Same interpretation when sample 16 is taken is applicable to sample 20. In addition, a univariate control chart is constructed for each quality characteristic as shown in Figure 3. However, results obtain from the univariate control charts performed poorly for its inability to detect an out of control variable. This is one of the advantages of the multivariate control charts over the univariate ones, its ability to detect small to moderate shift.

5.0 Concluding Remarks

This study introduced the bootstrap approach as a means of determining the control limits of a T^2control chart when the observations do not follow a normal distribution. The multivariate quality control has become important as the growth of technology has made it relatively easy to monitor many quality characteristics on each unit of product manufactured. In practice, there are not a few situations in which the simultaneous monitoring and control of two or more related quality characteristics is necessary. The decomposition approach of identifying out of control signal has been introduced. Using a numerical example, the effectiveness of this approach has been shown. Therefore, the values of bootstrap control limit (CL_B) and decomposition (d_i) obtained in this study will assist the management to take decision for monitoring future production purposes.

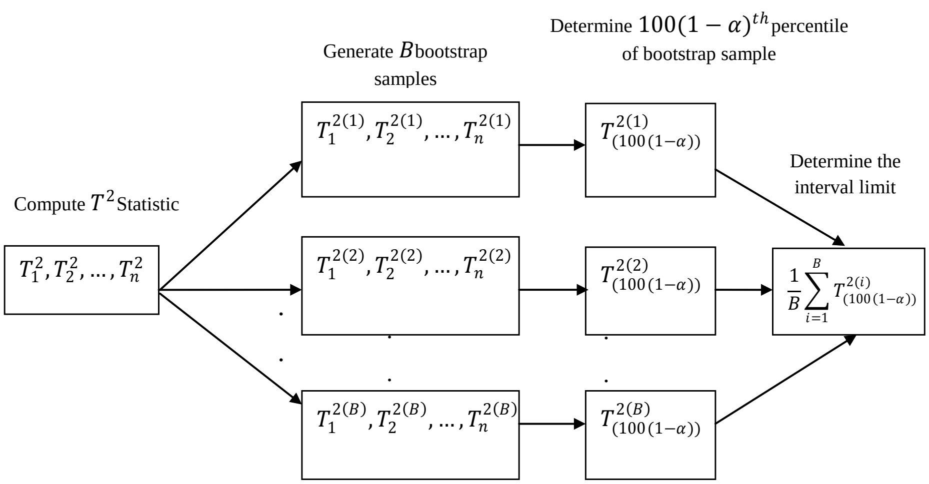

Fig. 1: Flowchart showing Bootstrap Procedure for calculating Control limits

Table 1: Two observable quality characteristics from a production process.

Sample	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x_1	58	60	50	54	63	53	42	55	46	50	49	57	58	75	55
x_2	32	33	27	31	38	30	20	31	25	29	27	30	33	45	27

Table 2: Sample's Representative of Production Process

Sample	x_1	x_2	T^2
1	58	32	1.126761
2	60	33	3.169014
3	50	27	3.169014
4	54	31	2.042254
5	63	38	13.52113
6	53	30	1.690141
7	42	20	22.8169
8	55	31	0.704225
9	46	25	10.6338
10	50	29	6.690141
11	49	27	5.070423
12	57	30	1.690141
13	58	33	1.901408
14	75	45	52.8169
15	55	27	6.338028

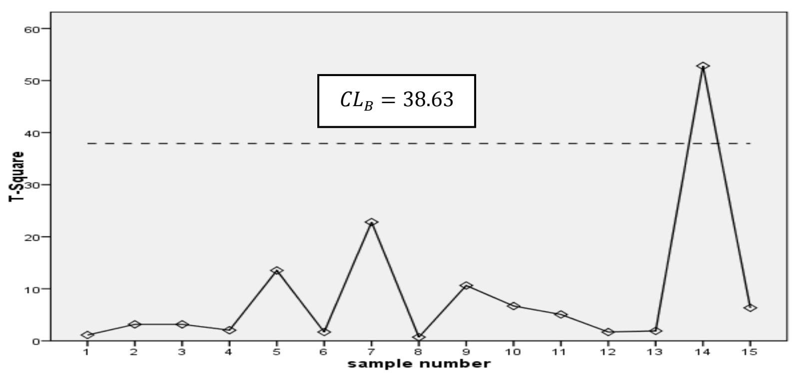

Figure 2: Multivariate Hotelling'sT^2 control chart

Table 2: Additional Sample Representative of Production Process

Sample	x_1	x_2	T^2
16	60	26	40.14085
17	43	28	41.69014
18	58	29	7.253521
19	66	43	39.22535
20	52	36	45.6338

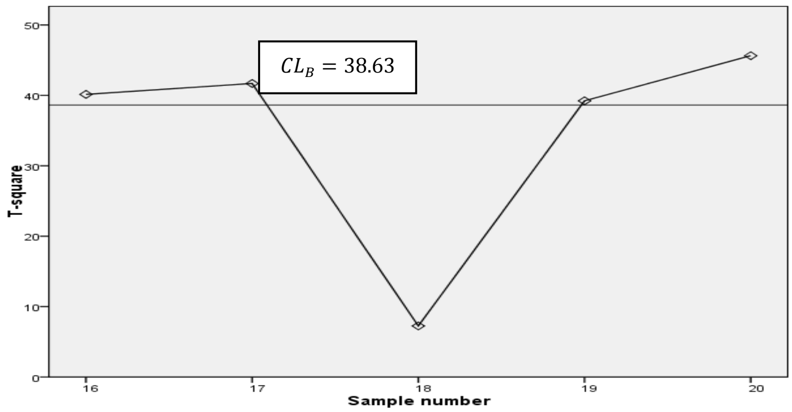

Figure 3: Multivariate Hotelling'sT^2 control charts for additional data

Table 4: Out-of-Control Signal's Interpretation

Sample	T^2	T_1^2	T_2^2	d_1	d_2
16	40.14085	3.333333	3.125	36.80751	37.01585
17	41.69014	0.833333	18	40.85681	23.69014
18	7.253521	0.208333	1.125	7.045188	6.128521
19	39.22535	35.20833	15.125	4.017019	24.10035
20	45.6338	7.5	1.125	38.1338	44.5088

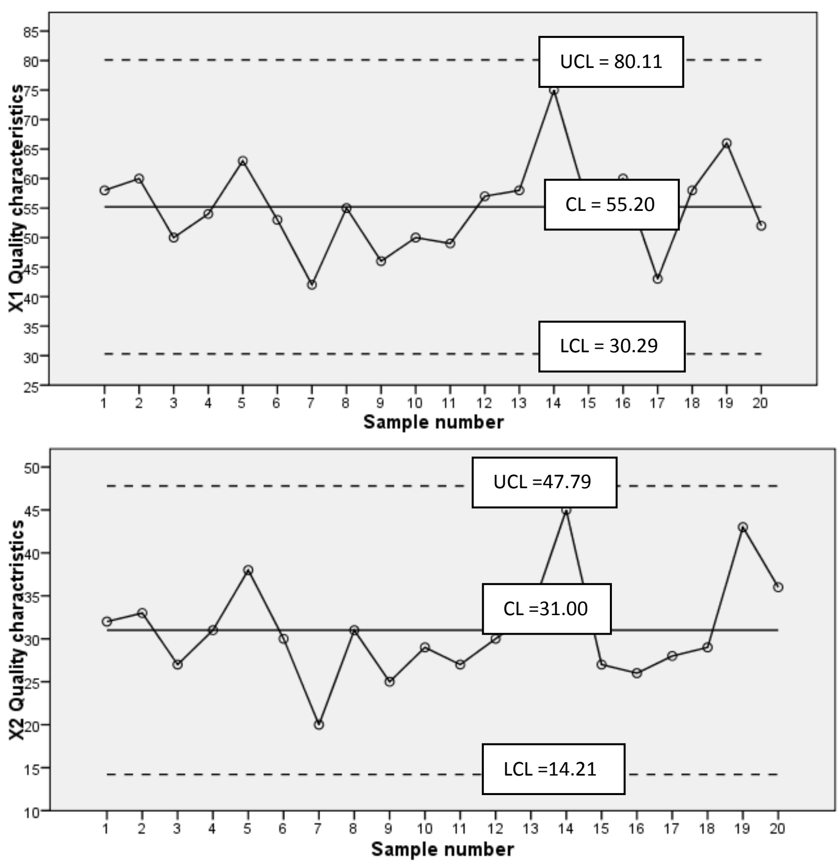

Figure 3: Univariate control charts

6.0 References

[1] Woodall W.H. and Montgomery D.C. (1999). Research Issues and Ideas in Statistical Process Control, Journal of Quality Technology, 31:376-386

[2] Alt, F.B. (1985): Multivariate Quality Control. The Encyclopedia of Statistical Sciences, Kotz S., Johnson N.L., and Read C.R., Eds., John Wiley and Sons, New York, 110-122

[3] Chua M.K. and Montgomery D.C. (1992). Investigation and Characterization of a Control Scheme for Multivariate Quality Control, Quality and Reliability Engineering International, 8: 37- 44

[4] Hayter A.J and Tsui K.L. (1994): Identification and Quantification in Multivariate Quality Problems. Journal of Quality Technology, 26:197-208

[5a] Mason R.L., Tracy N. D. and Young J.C. (1995).Decomposition of Multivariate Control Chart Interpretation. Journal of Quality Technology, 27: 99-108.

[5b] Mason R.L., Tracy N. D. and Young J.C. (1997). A Practical Approach for Interpreting Multivariate Control Chart Signals, Journal of Quality Technology, 29: 396-406

[6] Hasssan A, Baksh M.S., Shaharoun A.M and Jamaluddin H.(2003): Improved SPC Chart Pattern Recongnition using Statistical Features. International Journal of Production Research; 41(7): 1587-1603

[7] Montgomery D.C. (2005). Statistical Quality Control, 5th ed., John Wiley and Sons, New Jersey

[8] Taleb H., Limam M. and Hirota K. (2006): Multivariate Fuzzy Multinomial Control Charts, Quality Technology and Quantitative Management, 3(4): 437-453

[9] Bersimis S., Psarakis S. and Panaretos J. (2007). Multivariate Statistical Process Control Charts, Quality and Reliability Engineering International, 23: 517 – 543

[10] Guh R. (2007): On Line Identification and Qualification of Mean Shifts in Bivariate Process using a Neural Network based Approach; Quality and Reliability Engineering International, 23(3): 367- 385

[11] Li T., Hu S., Wei Z. and Liao Z. (2013): A Framework for Diagnosing the out of Control Signals in Multivariate Process using Optimized Support Vector Machines. Mathematical Problems in Engineering in http://dx.doi.org/10.1155/2013/494626.

[12] Chatterjee S. and Qiu P. (2009): Distribution-Free Cumulative Sum Control Charts using Bootstrap Based Control Limits. The Annals of Applied Statistics, 3(1) 349-369

[13] Phaladiganon P., Kim S.B., Chen V.C.P., Baek J.G. and Park S.K. (2011): Bootstrap Based T^2 Multivariate Control Charts. Communication in Statistics-Simulation and Computation, 40(5): 645-662

[14] Jackson J.E. (1991): A user guide to principal components; John Wiley and Sons: New York

[15] Marvelakis P.E., Bersimis S., Panaretos J. and Psarakis S. (2002): Identifying the out of Control Variable in a Multivariate Control Chart.Communication in Statistics: Theory and Methods, 31(12):2391-2408

[16] Liu R.Y. and Tang J. (1996): Control Charts for Dependent and Independent Measurements Based on the Bootstrap. Journal of American Statistical Association; 91(436): 1694-1700

[17] Jones I.A. and Woodall W.H. (1998): The Performance of Bootstrap Control Charts. Journal of Quality Technology; 30(4): 362-375

[18] Lio Y.L. and Park C. (2008): Bootstrap Control Chart for Birnbaum-Saunders Percentiles. Quality and Reliability Engineering International; 24(5): 585-600

[19] Park H.I. (2009): Median Control Charts Based on Bootstrap Method. Communication in Statistics-Simulation and Computation, 38(1): 558-570

[20] Adewara J.A. and Adekeye K.S. (2012): Bootstrap T Technique for Minimax Multivariate Control Chart. Open Journal of Statistics, 2 469-473.

[21] Mason R.L. and Young J.C. (2002). Multivariate Statistical Process Control with Industrial Applications, ASA-Society for Industrial and Applied Mathematics, Philadelphia.

[22] Efron B. (1979): Bootstrap Methods; Another look at the Jackknife. The Annals of Statistics, 7, 1-26

[23] Efron B. (1982): The Jackknife, the Bootstrap and other Resampling Plans. Society of Industrial and Applied Mathematics, CBMS-NSF Monographs, 38.

[24] Efron B. and Tibshirani R.J.(1993): An introduction to the bootstrap. New York; Chapman and Hall.

[25] Runger G.C., Frank A.B and Montgomery D.C. (1996): Contributors to a Multivariate Statistical Process Control Chart Signal, Communication in Statistics: Theory and Methods, 33:299-313