Journal of the Nigerian Association of Mathematical Physics
Volume26, (March, 2014), pp 294 – 298

Control of population flow of a Markov Chain

Lawan Adamu Isma'il

Department of Statistics, Kano University of Science and Technology, Wudil

Abstract

This paper is concerned with realization of the model of Markov population flow as the number of steps increases when the system's capacity η. Students' population flow was considered to make realization of the model under the assumed system's capacity. I realize the model gives as output what was inputted at initial, first and second states of the process, cumulative number of students who remain with those absorbed from the system and finally achieved the stationary input to the system as the number of steps goes large.

Keywords: Markov Chain, Markov Matrix, State Vector

1.0 Introduction

Shafiqah and Mokhtar [1], presented an application of Markov analysis of student flow in a higher educational institute using historical data of a random sample of 250 students of Faculty of Science at Kuwait University. They discovered as follows;

i. A freshman student has about 0.61 probability of graduating
ii. Freshman, sophomore, junior students stay on the average of 3, 2.4, 2.26 semesters at their respective levels before they pass on to the next level of study, while senior students stay longer, on average of 3.74 semesters
iii. A high percentage of 39% of incoming freshman students withdraw from their study and
iv. The probability of progression to a higher level and graduating increases as students move on to a higher level in the system.

Wolfgang S. [2], considered a classical population flow model in which individuals pass through n strata with certain state-dependent probabilities and at every time $t = 0, 1, 2, \ldots\ldots,$ there is a stochastic inflow of a new individuals to every stratum. He proved for a stationary inflow process the convergence of the joint distribution of group sizes and derives the limiting Laplace transform.

He considered a population whose members are divided into n ordered strata $S_1, S_2 \ldots S_n$.

At discrete times $t = 1, 2\ldots$ new individuals enter the strata; let $X_i(t)$ be the number of individuals entering S_i from outside at time t. He assumed that

$X(t) = (X_1(t), \ldots\ldots, X_n(t))^T$, $t \in$ □

as an n-dimensional, $□^n_+$-valued, strictly stationary stochastic sequence. From time t to time $t+1$ $(t = 0, 1, 2, \ldots\ldots)$ each individual either stays in its current stratum, moves to its successor, or leaves the system, which he modeled by introducing an additional stratum S_{n+1} with probability r_i, where $p_i, q_i, r_i \geq 0$ and $p_i + q_i + r_i = 1$, $i = 1, 2, \ldots\ldots, n,$ and $i = n$ we set $p_i = 0$. All individuals are assumed to independently of each other.

Kneale and Robert [3],propose a simple attendance and enrollment model based on the "Constant-work" that is required of students before they can obtain a degree.

The probability model they proposed is simple in several respects: it assumes that, except for a quantitative measure of work completed toward a degree, a student make a decision to attend, vacation, or drop out with probabilities p, q, *and* r respectively, such that $p + q + r = 1$. These probabilities are independent of time or whether the student is freshman, sophomore, junior, or senior. If a student attends the university, the conditional probability that he will complete one unit out

Corresponding author: E-mail: abbansta@yahoo.com, Tel.: +2348064955053

of a total of *w* units of work is *s;* the conditional probability that he fails to complete a unit of work in the semester is *1-s*.

In this paper, however, the data of Students' transition from one level to another (i.e from Level 400 to Spill over I, II and finally to Graduation or Withdrawal) of the Department of Statistics of Kano University of Science and Technology, Wudil, is used and evaluated into Markov matrix following the procedure described in Anderson and Goodman [4].

It is assumed as the model that a student independently stays in a particular level L_i with probability q_i, moves to the next immediate L_{i+1} with probability p_i or leaves the system $\left(L_{n+1}\right)$ with probability r_i.

The analysis of continuous-time Markov chains (CTMCs) is similar to that of discrete-time case, except that the transitions from a given state to another can take place at any instant of time. This implies that, although the parameter *t* has a continuous range of values, the set of values of $X\left(t\right)$ is discrete as described by Kishor [5].

The purpose of this paper is not only to test the validity of the model but to check the possible deficiency with view of remedying it and to further check what can be extracted out from the model.

Theorem [6]

If *P* is an $m\times m$ stochastic transition matrix and $Y^{(0)}=(y_1^{(0)},y_2^{(0)},...,y_m^{(0)})$ is the state vector containing the number of units in each of the *m* strata of the system at initial step of the process, then the expected number of units after *n* steps is given by

$$X^{(n)}=\begin{cases} Y^{(0)}P; & n=1 \\ Y^{(0)}P^{(n)}+\sum_{j=1}^{n-1}Y_1^{(n-j)}P^{(J)}; & n=2,3,..\ n>1 \end{cases} \tag{1}$$

Suppose that new inputs are allowed to enter the system at constant rate at beginning of each step of the process. Then the state vector after *n*-steps is given by

$$X^{(n)}=Y^{(0)}P^{(n)}+Y\sum_{j=1}^{n-1}P^{(j)}=X_a^{(n)};n>1 \tag{2}$$

Then, if the system is restricted to accommodate a total number of η units, then we can determine the expected number of units $Y=\left[\frac{\eta-Y^{(0)}P^{(0)}\theta}{\sum_{i=1}^{m-1}\alpha_{1i}},0,...,0\right]$ that could be admitted into the system at each step, so that the system's capacity is sustained at the neighborhood of η.

2.0 Data Analysis

The analysis will be done using MATLAB and MATHEMATICA soft wares.

If at initial state there fifty students, then the state vector with five levels as states is

$$Y^{(0)}=\left(50,0,0,0,0\right)=X^{(0)}$$

The students' Markov matrix is obtained as

$$P=\begin{bmatrix} 0 & 0.578 & 0 & 0 & 0.422 \\ 0 & 0 & 0.396 & 0.018 & 0.586 \\ 0 & 0 & 0 & 0.146 & 0.586 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$X^{(1)}=Y^{(0)}P$ for $n=1$

$$Y^{(0)}P=(50,0,0,0,0)\begin{bmatrix}0 & 0.578 & 0 & 0 & 0.422\\ 0 & 0 & 0.396 & 0.018 & 0.586\\ 0 & 0 & 0 & 0.146 & 0.854\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}=X^{(1)}=(0,28.9,0,21.1)$$

Define

$$\sum_{j=1}^{4}P^{j}=P^{1}+P^{2}+P^{3}+P^{4}=\begin{bmatrix}0 & 0.578 & 0.2289 & 0.0980 & 3.0951\\ 0 & 0 & 0.396 & 0.2454 & 3.3586\\ 0 & 0 & 0 & 0.584 & 3.3586\\ 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 0 & 4\end{bmatrix}$$

Using

$$Y=\left(\frac{\eta-Y^{(0)}P^{(n)}\theta}{\sum_{i=1}^{m-1}\alpha_{1i}},0,\ldots,0\right),$$

We can obtain constant values that could be admitted into the system after each step of the process for the corresponding system capacity η

Now,

$$\sum_{i=1}^{m-1}\alpha_{1i}=(0.9049)$$

Assuming different values of η we have;

where $n=2,3,4,5$

Case 1 with $\eta=65$, we have

$$X^{(n)}=Y^{(0)}P^{(n)}+Y\sum_{j=1}^{n-1}P^{(j)}=X_a^{(n)}\quad ;n>1$$

Using constant inputs obtained from the number of steps for the corresponding system capacity we have

$$Y=(58.61,0,0,0,0),(69.41,0,0,0,0),(69.41,0,0,0,0),(69.41,0,0,0,0)$$

$$X^{(2)}=(50,0,0,0,0)P^{2}+(58.61,0,0,0,0)P=(0,33.88,11.44,0.52,62.77)$$

$$X^{(3)}=(50,0,0,0,0)P^{3}+(69.41,0,0,0,0)\left(P^{1}+P^{2}\right)=(0,40.12,15.89,2.91,129.90)$$

$$X^{(4)}=(50,0,0,0,0)P^{4}+(69.41,0,0,0,0)\left(P^{1}+P^{2}+P^{3}\right)=(0,40.12,15.89,5.95,196.27)$$

$$X^{(5)}=(50,0,0,0,0)P^{5}+(69.41,0,0,0,0)\left(P^{1}+P^{2}+P^{3}+P^{4}\right)=(0,40.12,15.89,9.00,262.64)$$

Case 2 with $n=85$, we have

$$X^{(n)}=Y^{(0)}P^{(n)}+Y\sum_{j=1}^{n-1}P^{(j)}=X_a^{(n)}\quad ;n>1$$

Using constant inputs obtained from the number of steps for the corresponding system capacity we have

$$Y = (80.71, 0, 0, 0, 0), (91.51, 0, 0, 0, 0), (91.51, 0, 0, 0, 0), (91.51, 0, 0, 0, 0)$$

$$X^{(2)} = (50, 0, 0, 0, 0) P^2 + (80.71, 0, 0, 0, 0) P = (0, 46.65, 11.44, 0.52, 72.10)$$

$$X^{(3)} = (50, 0, 0, 0, 0) P^3 + (91.51, 0, 0, 0, 0)\left(P^1 + P^2\right) = (0, 52.89, 20.95, 3.14, 156.04)$$

$$X^{(4)} = (50, 0, 0, 0, 0) P^4 + (91.51, 0, 0, 0, 0)\left(P^1 + P^2 + P^3\right) = (0, 52.89, 20.95, 7.15, 243.54)$$

$$X^{(5)} = (50, 0, 0, 0, 0) P^5 + (91.51, 0, 0, 0, 0)\left(P^1 + P^2 + P^3 + P^4\right) = (0, 52.89, 20.95, 11.16, 331.04)$$

Case 3 with $\eta = 100$, we have

$$X^{(n)} = Y^{(0)} P^{(n)} + Y \sum_{j=1}^{n-1} P^{(j)} = X_a^{(n)} \ ; n > 1$$

Using constant inputs obtained from the number of steps for the corresponding system capacity we have

$$Y = (97.29, 0, 0, 0, 0), (108.08, 0, 0, 0, 0), (108.08, 0, 0, 0, 0), (108.08, 0, 0, 0, 0)$$

$$X^{(2)} = (50, 0, 0, 0, 0) P^2 + (97.29, 0, 0, 0, 0) P = (0, 56.23, 11.44, 0.52, 79.09)$$

$$X^{(3)} = (50, 0, 0, 0, 0) P^3 + (108.08, 0, 0, 0, 0)\left(P^1 + P^2\right) = (0, 62.47, 24.74, 3.32, 175.64)$$

$$X^{(4)} = (50, 0, 0, 0, 0) P^4 + (108.08, 0, 0, 0, 0)\left(P^1 + P^2 + P^3\right) = (0, 62.47, 24.74, 8.05, 279.00)$$

$$X^{(5)} = (50, 0, 0, 0, 0) P^5 + (108.08, 0, 0, 0, 0)\left(P^1 + P^2 + P^3 + P^4\right) = (0, 62.47, 24.74, 12.79, 382.32)$$

3.0 Discussions of Result

From Table 1, it can be observed that the model after three steps of Regular, Spill over I and Spill over II denoted by $n = 0, 1, 2$ respectivelygives out exactly what was inputted at initial states. This is because there are only three maximum steps remaining for the student to be absorbed.

For the remaining three steps, the model only gives the number that remains in the system and maintained the constant input after third step. This is because all steps after the third are absorbing steps.

4.0 Limitations of the Model

The model is more suitable to non Singular Matrix and limited to only number of years that a student will stay on and after regular level 400.

The model for limiting distribution is only applicable to non Singular Matrix

The model does not take into consideration the students that might be absorbed (withdrawn) at Level 400 having been under probation at Level 300.

The model does not also take into consideration students that might be expelled from the system due to examination malpractice and other possible reasons.

5.0 Conclusion

In each of the above three selected system's capacity η with the corresponding number of steps *n*, it is realized that the model gives the cumulative number of students who remain in the system with those absorbed.

The transitional probability can also be obtained from the model; hence the probable number of students moving from one state to another in an institution over a given period of time can also be predicted using the model.

Finally, the model is observed to be practically okay as well.

TABLE 1: THREE ASSUMED CASES OF SYSTEM'S CAPACITY

η	n	$Y^{(0)}P^{(n)}\theta$	$\eta - Y^{(0)}P^{(n)}\theta$	$\sum_{i=1}^{m-1}\alpha_{1i}$	Y	Population in system	Population Absorbed
	0				(50,0,0,0,0)	50	
	1				(0,28.9,0,0,21.1)	29	21
65	2	11.96	53.04	0.9049	(0, 34, 11, 1, 62)	45	63
	3	2.19	62.81		(0, 40, 16, 3, 130)	56	133
	4	2.19	62.81		(0, 40, 16, 6, 196)	56	202
	5	2.19	62.81		(0, 40, 16, 9, 262)	56	271
	0				(50,0,0,0,0)	50	
	1				(0,28.9,0,0,21.1)	29	21
85	2	11.96	73.04	0.9049	(0, 47, 11, 1, 72)	58	73
	3	2.19	82.81		(0, 53, 21, 3, 156)	74	159
	4	2.19	82.81		(0, 53, 21, 7, 243)	74	250
	5	2.19	82.81		(0, 53, 21, 11, 331)	74	342
	0				(50,0,0,0,0)	50	
	1				(0,28.9,0,0,21.1,)	29	21
100	2	11.96	88.04	0.9049	(0, 56, 11, 1, 79)	68	80
	3	2.19	97.81		(0, 62, 25, 3, 176)	87	179
	4	2.19	97.81		(0, 62, 25, 8, 279)	87	287
	5	2.19	97.81		(0, 62, 25, 13, 382)	87	395

6.0 Refrences

[1] Shafiqah A. A. and Mokhtar K. (1998), An Application Of Absorbing Markov Analysis To The Student Flow In An Academic Institution. *The First Arab Statistical Conference*, (2007), pp. 560-581

[2] Wolfang S. (1999), Stationarity of a Stochastic population Flow. *Journal of applied probability*. Vol. 36, No. 1 (1999), pp. 291-294

[3] Kneale T.M. and Robert M.O. (1970), A Constant-Work Model for Student attendance and Enrollment. *Operations Research*, Vol. 18, No. 2 (1970), pp. 193-206

[4] Anderson T. W. and Goodman L. A. (1957), Statistical Inference About Markov Chain. *The Annals of Mathematical Statistics.* Vol. 28, No. 1 (1957), 81-110.

[5] Kishor S. T. (2005), Probability and Statistics with Reliability, Queuing and Computer Science Applications. *A John Wiley & Sons, inc*. pp. 228-392.

[6] Yakasai B.M. (2005), Stationary Population Flow of a Semi-open Markov Chain. *Journal of Nigerian Mathematical Physics.* Vol. 7, (2005), 395-398