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The simulation of a two dimensional Fokker Planck equation for an energetic alpha particle beam in a plasma using the Fokker Planck Package (FPPAC 81)

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Abstract

The two dimensional Fokker Planck equation is used to describe the time evolution of energetic beam of ions under the effect of dynamical friction and diffusion in velocity space, in a background plasma. The equation is simulated using the Fokker Planck package (FPPAC 81) to obtain the slowing down distribution function for energetic alpha particles which are produced continuously during deuterium-tritium fusion. Comparison was made with an analytical slowing down distribution function which ignores diffusion in pitch angle space. The slowing down of the alpha particle energy with time was also computed and a comparison was made to that of a 180 keV deuterium beam in a 5 keV plasma from literature.

Keywords: Dynamical Friction, Diffusion, Distribution function, Critical Velocity, Injection Current, Reaction Rate

1.0 Introduction

A situation that arises in many naturally occurring plasmas as well as fusion plasmas is that of a beam of fast ions moving through a plasma. The beam ion energy is typically much larger than the temperature of the background plasma, i.e the velocity of the beam V_b is greater than V the velocity of the background ion velocity, but less than V_e the background electron velocity. The beam ion may be the same type as the background ion or different.
In fusion research the plasma is self heated by the energetic ions produced during the fusion reactions themselves.

$$^{2}_{1}D + ^{3}_{1}T \rightarrow ^{4}_{2}He(3.51MeV) + ^{1}_{0}n(14.6MeV) \qquad (1)$$

The alpha particles produced in the reaction above have energy 200 times that of the background plasma ion and are born with an isotropic distribution of velocities, and thermalises with the background plasma particles as a result of multiple Coulomb collisions. In this work the thermalisation process is simulated by using the Fokker Planck Package (FPPAC81) to solve the Fokker Planck equation (FPE) for the beam ions, as it moves through the background plasma composed of Maxwellian ions and electrons with densities n_i,n_e, much greater than n_b, the beam ion density, in which Coulomb collisions of the beam ions with the background plasma results in the frictional drag on the background ions and electrons which cause the beam ions to slow down, and angular scattering on the background ions which cause the beam ions to be deflected from their original direction. Section 2 comprises of the Fokker Planck theory, section 3 describes the methodology, section 4 consists of results and discussion, and section 5 is the conclusion.

2.0 Theory

Collisional interactions in a fully ionized plasma are predominantly due to the cumulative effects of many small angle deflections rather than due to a few large angle deflections. A formulation for describing the effects of multiple small angle Coulomb collisions on the distribution function $f(\vec{v})$ is the Boltzmann's kinetic equation with a collision term, known as Fokker Planck equation[1].

$$\frac{\partial f_a}{\partial t} + \vec{v}\frac{\partial f_a}{\partial r} + \frac{F_a}{m_a}\frac{\partial f_a}{\partial v} = \left(\frac{\partial f_a}{\partial t}\right)_{coll} + S_a + L_a \qquad (2)$$

Where f_a, is the distribution function of the beam ion, F_a is the force, $\left(\frac{\partial f_a}{\partial t}\right)_{coll}$ is the collision term, S_a is the source term, and L_a the loss term.
The collision term is given by[2,3]

$$\left(\frac{\partial f_a}{\partial t}\right)_{coll} = -\frac{\partial}{\partial v}[\langle\Delta v\rangle f(v,t)] + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_k}[\langle\Delta v_i \Delta v_k\rangle f(v,t)] \qquad (3)$$

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Equation (3) was formulated by A.D. Fokker and M. Planck to treat Brownian motion [2,3]. It describes the evolution in time of the distribution function due Coulomb collisions. Where the magnitude of $\langle \Delta v \rangle$ is the dynamical friction giving rise to a slowing down of the directed motion of the particles, and $\langle \Delta v_i \Delta v_k \rangle$ are velocity diffusion coefficient's and their effect is to bring about spreading of the particles velocity over a wider region of velocity space. The competition between the dynamical friction and velocity diffusion gives rise to the Maxwellian distribution in steady state.
A modification of the formulation of the Fokker-Planck equation derived by Rosenbluth et al [3] is

$$\langle \Delta v \rangle_a = \Gamma_a \frac{\partial h_a}{\partial v}, \quad (4)$$

$$\langle \Delta v_i \Delta v_k \rangle_a = \Gamma_a \frac{\partial^2 g_a}{\partial v_i \partial v_k} \quad (5)$$

So that the Fokker Planck operator becomes

$$\left.\frac{\partial f}{\partial t}\right|_{Coll} = \Gamma_a \left\{ \partial / \partial v_i \left(f_a \frac{\partial h_a}{\partial v_i} \right) + \frac{1}{2} \frac{\partial^2}{\partial v_i \partial v_k} \left(f_a \frac{\partial^2 g_a}{\partial v_i \partial v_k} \right) \right\}, \quad (6)$$

Where the usual summation convention over repeated index i and j are used and where

$$\Gamma_a = \frac{4\pi Z_a{}^4 e^4}{m_a^2}. \quad (7)$$

The Rosenbluth potentials are

$$g_a = \sum_b \left(\frac{Z_b}{Z_a} \right)^2 ln\Lambda_{ab} \int f_b(v') \, |v - v'| dv' \quad (8)$$

$$h_a = \sum_b \frac{m_a + m_b}{m_b} \left(\frac{Z_b}{Z_a} \right)^2 ln\Lambda_{ab} \int f_b(v') \, |v - v'|^{-1} dv' \quad (9)$$

Here, the Coulomb logarithm, depends on both the interacting species, and is

$$\ln\Lambda_{ab} = \ln \left\{ \left(\frac{m_a m_b}{m_a + m_b} \right) \left(\frac{2\alpha c \lambda_D}{e^2} \right) \max \left(\frac{2E_k}{m_k} \right)_{a,b}^{1/2} \right\} - 1/2 \quad (10)$$

where $\alpha = 0.0073$ is the fine structure constant, $\lambda_D = \sqrt{\frac{E_e}{6\pi n_e^2}}$ is the Debye length
n_e is the electron density and E_e is the electron mean energy.
Equation (6) then becomes (in conservative form)

$$\frac{1}{\Gamma_a} \left(\frac{\partial f_a}{\partial t} \right) = \frac{1}{v^2} \frac{\partial G_a}{\partial v} + \frac{1}{v^2 \sin\theta} \frac{\partial H_a}{\partial \theta}, \quad (11)$$

Where

$$G_a = A_a f_a + B_a \frac{\partial f_a}{\partial v} + C_a \frac{\partial f_a}{\partial \theta}, \quad (12)$$

$$H_a = D_a f_a + E_a \frac{\partial f_a}{\partial v} + F_a \frac{\partial f_a}{\partial \theta}. \quad (13)$$

The coefficients $A_a, B_a, C_a, D_a, E_a, F_a$ are given in [3]
As suggested by Rosenbluth et al [3], the ''Rosenbluth potentials'' and the distribution functions themselves may be represented by expansions in the Legendre polynomials.

$$f_a(v, \theta, t) = \sum_{j=0}^{\infty} V_j^b(v, t) \, P_j(\cos\theta), \quad (14)$$

Where, $V_j^b(v, t) = \frac{2j+1}{2} \int_{-1}^{+1} f_b(v, \theta, t) \, P_j(\cos\theta) d(\cos\theta)$,
The expansions for the potentials are

$$g_a(v, \theta, t) = \sum_{j=0}^{\infty} \sum_b \left(\frac{Z_b}{Z_a} \right)^2 ln\Lambda_{ab} \, B_j^b(v, t) \, P_j(\cos\theta), \quad (15)$$

and

$$h_a(v, \theta, t) = \sum_{j=0}^{\infty} \sum_b \left(\frac{Z_b}{Z_a} \right)^2 \left(\frac{m_a + m_b}{m_b} \right) ln\Lambda_{ab} \, A_j^b(v, t) \, P_j(\cos\theta). \quad (16)$$

With coefficients given by,

$$A_j^b(v, t) = \frac{4\pi}{2j+1} \left[\int_0^v \frac{(v')^{j+2}}{v^{j-1}} V_j^b(v, t) dv' + \int_v^{\infty} \frac{v^j}{v'^{j-1}} V_j^b(v, t) dv' \right] \quad (17)$$

$$B_j^b(v, t) = -\frac{4\pi}{4j^2 - 1} \left[\int_0^v \frac{(v')^{j+2}}{v^{j-1}} \left(1 - \frac{j - \frac{1}{2}(v')^2}{j + \frac{3}{2}(v)^2} \right) V_j^b(v, t) dv' + \int_v^{\infty} \frac{v^j}{v'^{j-3}} \left(1 - \frac{j - \frac{1}{2}(v)^2}{j + \frac{3}{2}(v')^2} \right) V_j^b(v', t) dv' \right] \quad (18)$$

2.1 Integration

Trapezoidal integration is used, such that the density of species a is given by

$$n_a = 2\pi \int \int f(v, \theta) v^2 \sin\theta \, d\theta \, dv = 2\pi \sum_{i,j} a_j b_i f_a(v_j, \theta_i), \quad (19)$$

and the energy density of species is

$$n_a E_a = \pi m_a \sum_{i,j} a_j b_i v_j^2 f_a(v_j, \theta_i) \quad (20)$$

2.2 Spatial Differencing

The spatial derivatives are discretised using central finite difference as follows

$$\frac{\partial}{\partial v}(Af)_{i,j} \approx \left(A_{i,j+1}f_{i,j+1} - A_{i,j-1}f_{i,j-1}\right)/2\Delta v_j, \tag{21}$$

$$\frac{\partial}{\partial v}\left(B\frac{\partial}{\partial v}f\right)_{i,j} \approx \left\{\frac{B_{i,j+1/2}\left(f_{i,j+1}-f_{i,j}\right)}{\Delta v_{j+1/2}} - \frac{B_{i,j-1/2}\left(f_{i,j}-f_{i,j-1/2}\right)}{\Delta v_{j-1/2}}\right\}/\Delta v_j, \tag{22}$$

$$\frac{\partial}{\partial v}\left(C\frac{\partial}{\partial \theta}f\right)_{i,j} \approx \left\{\frac{C_{i,j+1}\left(f_{i+1,j+1}-f_{i-1,j}+1\right)}{2\Delta\theta_i} - \frac{B_{i,j-1}\left(f_{i+1,j-1}-f_{i-1,j-1}\right)}{2\Delta\theta_i}\right\}/2\Delta v_j, \tag{23}$$

Where

$\Delta v_{j\pm1/2} = \pm\left(v_{j+1} \pm v_j\right),\ \Delta v_j = \left(v_{j+1} - v_{j-1}\right)/2, \Delta\theta_i = \left(\theta_{i+1} - \theta_{i-1}\right)/2$,
$\Delta\theta_{i\pm1/2} = \pm(\theta_{i\pm1} - \mp\theta_i), B_{i,j\pm1/2} = \left(B_{i,j\pm1} + B_j\right)/2$
Similarly for θdirection.

2.3 Time Discretisation

The collision operator is time integrated using alternating direction implicit (ADI), or fully. The procedure for the (ADI) [4,5] is given as:

$$Q^n\left(\frac{f_{i,j}^{n+1/2}-f_{i,j}^n}{\Delta t/2}\right) = \frac{1}{v^2}\frac{\delta}{\delta v}\left(A^n f_{i,j}^{n+1/2} + B^n\frac{\delta f_{i,j}^{n+1/2}}{\delta v} + C^n\frac{\delta f_{i,j}^{n+1/2}}{\delta\theta}\right)$$
$$+ \frac{1}{v^2\sin\theta}\frac{\delta}{\delta\theta}\left(D^n f_{i,j}^n + E^n\frac{\delta f_{i,j}^n}{\delta v} + F^n\frac{\delta f_{i,j}^n}{\delta\theta}\right) + K^n f_{i,j}^{n+1/2} + J^n \tag{24}$$

$$Q^n\left(\frac{f_{i,j}^{n+1}-f_{i,j}^{n+1/2}}{\Delta t/2}\right) = \frac{1}{v^2}\frac{\delta}{\delta v}\left(A^n f_{i,j}^{n+1/2} + B^n\frac{\delta f_{i,j}^{n+1/2}}{\delta v} + C^n\frac{\delta f_{i,j}^{n+1/2}}{\delta\theta}\right)$$
$$+ \frac{1}{v^2\sin\theta}\frac{\delta}{\delta\theta}\left(D^n f_{i,j}^{n+1} + E^n\frac{\delta f_{i,j}^{n+1}}{\delta v} + F^n\frac{\delta f_{i,j}^{n+1}}{\delta\theta}\right) + K^n f_{i,j}^{n+1} + J^n \tag{25}$$

The difference equations(24),and (25) and the boundary conditions written in the tridiagonal form are

$$-\alpha_{i,j}^n f_{i,j+1}^{n+1/2} + \beta_{i,j}^n f_{ij}^{n+1/2} - \gamma_{i,j}^n f_{i,j-1}^{n+1/2} = \delta_{i,j}^n \tag{26}$$

$$-\varepsilon_{i,j}^n f_{i,j+1}^{n+1} + \mu_{i,j}^n f_{ij}^{n+1} - \nu_{i,j}^n f_{i,j-1}^{n+1} = \delta_{i,j}^n \tag{27}$$

Where $\alpha, \beta, \gamma, \varepsilon, \mu, \nu$ are known quantities. The procedure for solving eqs. (26) and (27) is the standard technique for solving tridiagonal systems[5].

2.4 Analytical Beam Distribution Function

For an energetic beam of particles that are heavier than the plasma ions and electrons they interact with, dynamical friction is the dominant process through which slowing down occurs, and the distribution function is given by [6,7]

$$f(V) = \frac{S\epsilon_0^2 M M_b}{n_e Z Z_b^2 e^4 ln\Lambda}\left(\frac{1}{1+\frac{V^3}{V_c^3}}\right) \tag{28}$$

Where$S = n_D n_T\langle\sigma V\rangle_{DT}$is the alpha particle injection current,

$V_c = \left[3^{\frac{1}{3}}Z^{\frac{1}{3}}\left(\frac{\pi}{2}\right)^{\frac{1}{6}}\right]\left[\frac{T_e}{\left(m^{\frac{1}{3}}M^{\frac{2}{3}}\right)}\right]^{\frac{1}{2}}$ is the critical velocity of the beam particle(the velocity at which the energy lost by the beam is shared equally by the plasma ions and electrons). n_D, n_T,are the deuterium and tritium ion densities, $\langle\sigma V\rangle_{DT}$ is the reaction rate parameter,n_e is the electron density.$Z and\ Z_b$are the charge numbers for the plasma and beam ions, $M and\ M_b$ are the mass numbers for the plasma and beam ions, m and T_e are the mass and temperature of the electron respectively.

2.5 Slowing Down Time

The slowing down time for an ion in the plasma at temperature T_e is given by [7]

$$\tau_s = 0.012\frac{(T_e[keV])^{3/2}M_b}{n_e[10^{20}m^{-3}]Z_b^2} \tag{29}$$

and the critical energy $W_c = 1/2\, m_b v_c^2 = 14.8 T_e M_b\left(\frac{1}{n_e}\frac{\sum_i n_i Z_i^2}{M_i}\right)^{2/3}$ (30)

3.0 Methodology

3.1 Fokker Planck Package

The package was written by McCoy et al [1] and solves the Fokker–Planck equations for an arbitrary number of charged species, described by distribution functions of speed v and pitch angle θ in the presence of arbitrary number of fixed Maxwellian species. It comprises of two main parts. The subroutine COEF which is used to compute the Fokker Planck coefficients,and subroutine XSWEEP which is used to time integrate the distribution functions using either implicit operator splitting or an alternating direction implicit (ADI) method. The driver calls the subroutine INITIAL which reads the input data, initialize certain package and driver variables and call several subroutines, such as SETPARS which copies values in parameter statement into common storage, XINIT for setting up the spatial mesh, XINITL for computing constants and constant arrays, FINIT which initializes the distribution functions, SOURCE for calculating the sources in velocity space, GNANDE for computing the densities, and energies. The initialization ends with a call to EEPRINT which print the distribution function and other information. The driver then solves the Fokker-Planck operator for NSTOP time steps, by calls to three package routines, SETITUP, COEF, and XSWEEP. Where, subroutine SETITUP in turn calls PREPKG1 for setting the boundary conditions, GNANDS for copying densities and energies into package arrays, while GAMMAI computes the Coulomb logarithm. After the new distribution are calculated, the new densities and energies are computed for all species in GNANDE. This is repeated for NSTOP time steps, and output is given for NPRINT times.

3.2 Procedure

An input file named alphabeam.dat was created, where an alpha particle of mass number M_b=4, charge number Z_b =2, injection energy $E_b = 3.50$ MeV, injection current $S = n_D n_T \langle\sigma V\rangle_{DT}$, where $\langle\sigma V\rangle_{DT} = 4.2 \times 10^{-22} m^3 s^{-1}$,was injected into a D-T plasma with $n_T = n_D = 5 \times 10^{19} m^{-3}$, $n_e = 10^{20} m^{-3}$,$T_e = T_i = 10, 15,\ and\ 20\ keV$. The ion species in the plasma was considered to be the average of M_D and M_Ti.e $M_{DT} = 2.5$, $Z_{DT} = 1.0$, $n_{DT} = 10^{20} m^{-3}$. The slowing down distribution for the energetic alpha particle beam was obtained from the FPPAC. The distribution function for the alpha particle was also obtained from the distribution given in Equation (28). A comparison was made between the two distributions at different values of T_e. The Slowing down of energy of the alpha particle beam in the plasma at $T_e = T_i = 10, 15,\ and\ 20\ keV$ was also simulated. A comparison was made to the slowing down of a 180 keV deuterium beam in a 5 keV plasma obtained from the literature [6]

4.0 Results

4.1 Slowing down distribution function

The graph of the slowing down distribution of energetic alpha particle beam as a function of normalized velocity is shown in figures 1 -3 for varying plasma temperatures T= 20keV-10keV. The critical velocity below which the energetic alpha particle beam heats the plasma ions only is seen to decrease with temperature. The computed distribution function which did not neglect angular dependence compares fairly well with the calculated slowing down distribution which neglects the pitch angle dependence[6,7]

4.2 Slowing down of Energy

Using FPPAC the slowdown of 3.5 MeV alpha particle energy with time (normalised) to density was computed for plasma temperatures T_e=10 -20 keV

4.3 Critical Energy and Slowing Down Times FOR 3.5 MeV AND A 180 KeV Deuterium Beam

5.0 Discussion/Conclusion

The energetic alpha particle beam was simulated using the Fokker Planck Package FPPAC81. Fig 1 for the variation of the distribution functionwith normalised velocity shows that as the energetic beam slows down in the plasma losing heat, the population of thermalized particles increases and this is consistent with Fig. 2 for the distribution obtained analytically. The graphs of energy slow down with time in Figs. 3-5 show that as the temperature of the plasma increases from 10-20 KeV the slowdown of energy becomes more effective, this as a result of the increase in the critical energy W_c which implies more energy is transferred to the ions in the plasma than the electrons. Table 1 above also shows the variation critical energy and the slowing down time with temperature with both increasing. The ratio of critical energy to beam energy is seen to increase with temperature and is more for the 180 KeV deuterium beam than for the alpha particle beam. This indicates that the rate of transfer of energy to ions in the plasma by the deuterium beam is more than that of the alpha particle beam at the same temperature.Figure 6 shows the slowdown of a 180 KeV beam in a tritium plasma at T_e=5 KeV which compares favourably with that of Figure 5 for the D-T plasma.

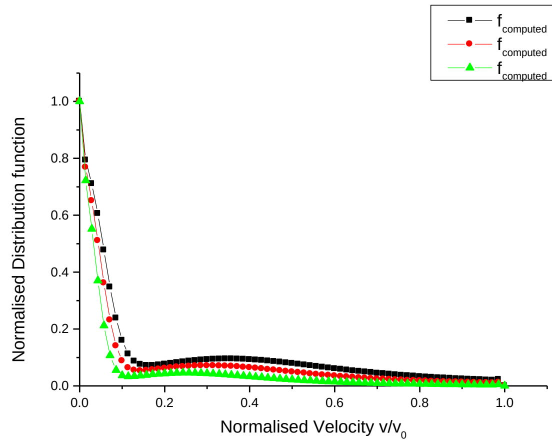

Fig 1: The normalised alpha particle beam Slowing down distribution functions, for Plasma at T=10, 15, and 20 keV. Using FPPAC.

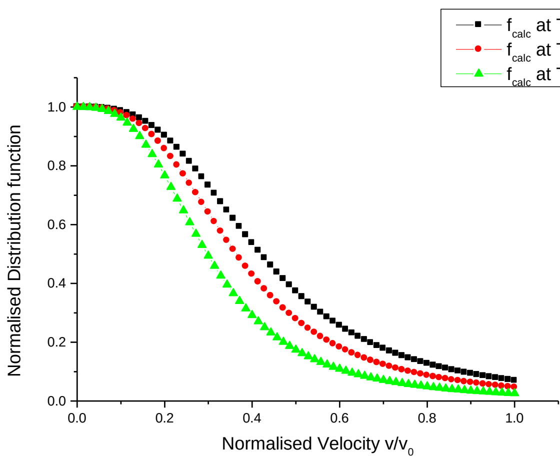

Fig 2: The calculated normalised alpha particle beam Slowing down distribution functions, for Plasma at T=10, 15, and 20 keV. Using Equation (28)

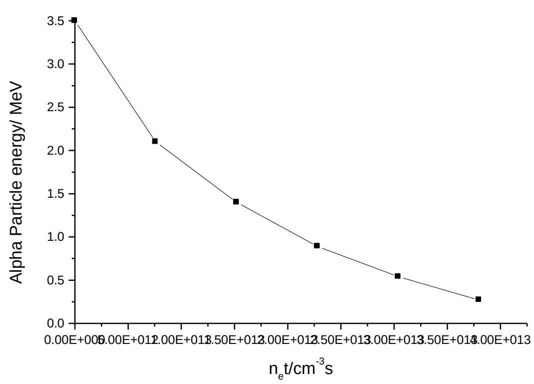

Figure3: The Slowing down of a 3.5 MeV energetic alpha particle beam in a D-T plasma at T=10 keV

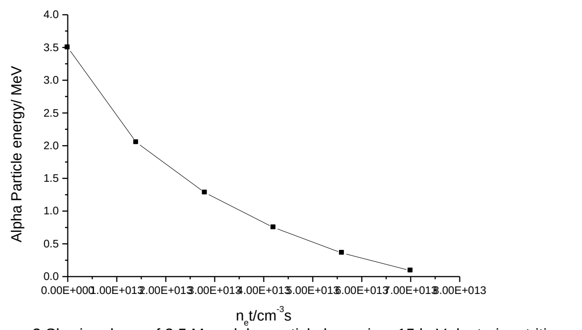

Figure 4: The Slowing down of a 3.5 MeV energetic alpha particle beam in a D-T plasma at T=15 keV

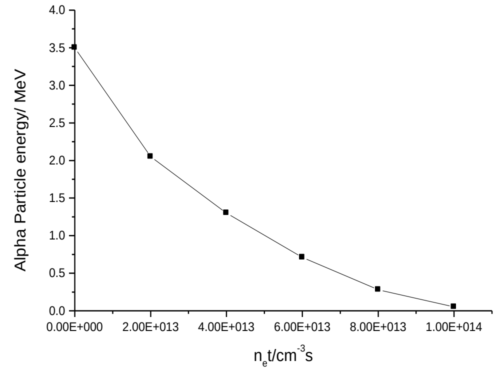

Figure 5: The Slowing down of a 3.5 MeV energetic alpha particle beam in a D-T plasma at T=20 keV

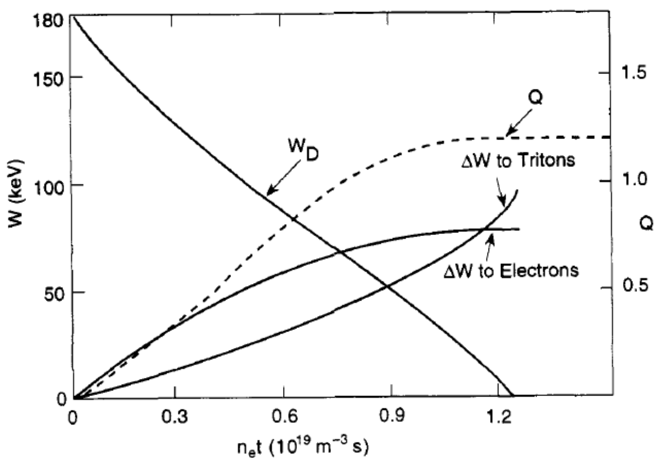

Figure 6: The Slowing down of a 180 keV deuteron injected into a tritium plasma with T = 5 keV. The energy of the deuteron is W_D and the energy increment,ΔW are given to the Tritium and electrons in the plasma [6]

Table 1: Critical energy and Slowing Down Times for 3.5 Mev and a 180 KeV Deuterium Beam

T_e/ keV	W_c	τ_s / ms	$(W_c/W_b)^{3/2}$
10	321.58	0.200	0.03
15	482.38	0.500	0.05
20	643.17	0.700	0.08
5	71.20	0.268	0.25

References

[1] McCoy, M. G., Mirin, A.A., Killeen, J., A Two–Dimensional Multispecies Nonlinear FokkerPlanck Package, Comp. Phys. Commun. **24,** 37-61 (1981)

[2] Chandrasekhar, S. Review Mod. Phys. **18**11(1943)

[3] Rosenbluth, M. N., MacDonald, W. M., Judd, D. L., Phys. Rev. **107**, 1 (1957).

[4] Press, W. H., Flannery, B. P., Teukolosky, S. A. andVetterlihg, W. T., "Numerical Recipes, The art of Scientific Computing " (CambridgeUniv. Press,1992).

[5] Ritchmeyer, R. D. and Morton, K. W., Difference methods for initial value problems (Wiley, New York, 1967).

[6] Goldston R.J. Rutherford H.P "Introduction to Plasma Physics" IOP Publishing Bristol and Philadelpia 1995.

[7] Koch R Plasma Heating by Neutral Beam Injection Laboratory for Plasma Physics Ecole Royal Militaire, Belgium(2000).