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Dynamical Analysis of a Finite Simply Supported Uniform Rayleigh Beam Under Travelling Distributed Loads

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Abstract

This paper investigates the dynamic behaviour of a finite uniform Rayleigh beam subjected to travelling distributed loads. The fourth order partial differential equation governing the motion of the beam is first treated with the Fourier Sine integral transformation to reduce it to a second order coupled ordinary differential equation which is further simplified using the modified asymptotic method of struble. The closed form solution obtained is analysed and numerical calculation and representation in plotted curves show that as the foundation modulus and rotatory inertia correction factor increase, the response amplitude of the dynamical system decreases. It is further deduced that, for the same natural frequency, the critical speed for the system traversed by a distributed force is greater than that traversed by moving distributed mass. Thus, resonance is reached earlier in the moving distributed mass system than in the moving distributed force system. This clearly shows that the moving distributed force solution is not an upper bound to the moving distributed mass problem.

1.0 Introduction

The interaction of travelling subsystems and structural systems is a major research area in the field of structural engineering design and construction. It is especially of enormous importance to the study of the dynamic characteristic of bridges or concrete slabs under moving masses. Thus, the subject has drawn considerable attention from researchers in Engineering and Physical Sciences [1,2,3] for well over a century. Most of the early previous analysis works in this area were directed at the dynamic behaviour of structures under the moving force. These include the work of Willis [4], Yoshida [5], Krylov [6] and Steel [7]. However, in the analysis of the effects of vehicles moving over large-span bridges, Inglis [8] introduced a theory according to where the gravitational effects of the moving load may be separated from the inertia ones. The separation is significant especially when the load mass around the beam mass are of considerable magnitude. An attempt to solve this type of dynamical problem was first made by Saller [9], then Jeffcott [10] whose iterative methods become divergent in some cases. The work of Stanisic et al [11] gave impetus to a wide range of analytical developments in this area of study in the recent years. Recent research contributions in this area of study include those of Mofid and Akin [12], Green and Cebon [13], Akin and Mofid [14], Yavari et al [15], Nikkhoo et al [16], Oni and Omolofe [17], Oni and Awodola [18], Oni and Omolofe [19]. Impressive though these works are, the authors simplified their investigations by modelling their loads as concentrated line loads which are mere approximation and not accurate representation of the load as it traverses the structure. To this end, this paper concerns the flexural motions of a finite simply supported uniform Rayleigh beam under travelling distributed masses. The focus is on analytical development so as to treat the issue of resonance phenomena. The influence of the various Rayleigh beam-structure parameters on the vibrating system shall be classified.

2.0 Mathematical Formulation

Consider the problem of the flexural vibrations of a finite uniform Rayleigh beam resting on an elastic foundation. The distributed load traversing the beam is assumed to move at uniform velocity. The equation of motion of the beam is the fourth order partial differential equation given by [1]

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$$EI\frac{\partial^4 V(x,t)}{\partial x^4}+\mu\frac{\partial^2 V(x,t)}{\partial t^2}-\mu r^0\frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}+KV(x,t)=P(x,t) \quad (2.1)$$

where x is the spatial coordinate, t is the time, $V(x,t)$ is the transverse displacement, E is Young's modulus, I is the constant Moment of inertia of the beam, μ is the constant mass per unit length of the beam, r^0 is the measure of rotatory inertia correction factor, K is the elastic foundation constant and $P(x,t)$ is the uniform distributed load acting on the beam. For this problem, the distributed load moving on the beam under consideration has mass commensurable with the mass of the beam. Consequently, the load inertia is not negligible but significantly affects the behaviour of the dynamical system. Thus, the distributed load $P(x,t)$ takes the form,

$$P(x,t)=P_f(x,t)\left[1-\frac{1}{g}\frac{d^2V(x,t)}{dt^2}\right] \quad (2.2)$$

where $P_f(x,t)$ is the continuous moving force acting on the beam model given by

$$P_f(x,t)=MgH(x-ct) \quad (2.3)$$

where c is the velocity of the distributed mass, the time t is assumed to be limited to that interval of time within which the mass μ is on the beam, that is

$$0\le ct\le L \quad (2.4)$$

g is the acceleration due to gravity, and $H(x-ct)$ is the Heaviside function defined as

$$H(x-ct)=\begin{cases}0, & x<ct\\ 1, & x>ct\end{cases} \quad (2.5)$$

$\frac{d^2}{dt^2}$ is the convective acceleration operator defined as

$$\frac{d^2}{dt^2}=\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2} \quad (2.12)$$

Substituting (2.2), (2.3) and (2.12) into (2.1), we obtain

$$EI\frac{\partial^4 V(x,t)}{\partial x^4}+\mu\frac{\partial^2 V(x,t)}{\partial t^2}-\mu r^0\frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}+KV(x,t)+$$

$$MH(x-ct)\left(\frac{\partial^2 V(x,t)}{\partial t^2}+2c\frac{\partial^2 V(x,t)}{\partial x\partial t}+c^2\frac{\partial^2 V(x,t)}{\partial x^2}\right)=MgH(x-ct)$$

(2.13)

Equation (2.13) is the fourth order partial differential equation governing the transverse displacement response of a uniform Rayleigh beam on a winkler elastic foundation and under the actions of moving distributed loads. The beam has span L and is simply supported. Accordingly, the deflections and moments vanish at end x = 0 and x = L_x. Thus,

$$V(x,0)=0=V(L,t) \quad (2.14)$$

and

$$\frac{\partial^2 V(0,t)}{\partial x^2}=0=\frac{\partial^2 V(L,t)}{\partial x^2} \quad (2.15)$$

The initial conditions are taken to be, without any loss of generality,

$$V(x,0)=0=\frac{\partial^2 V(x,0,)}{\partial x^2} \quad (2.16)$$

3.0 Solution Techniques

In this paper, in order to compute the transverse displacement $V(x,t)$ of the vibrating beam, use is made of the Finite fourier integral sine transform. This integral transformation technique is given by

$$\bar{V}(m,t)=\int_0^L V(x,t)\,Sin\frac{m\pi x}{L}dx \quad (3.1)$$

with inverse

$$V(x,t)=\frac{1}{L}\sum_{m=1}^{\infty}\bar{V}(m,t)Sin\frac{m\pi x}{L} \quad (3.2)$$

Applying the integral transform (3.1), the governing equation (2.13) becomes

$$EIT_A(t)+\mu\bar{V}_{tt}(m,t)-\mu r^0 T_B(t)+K\bar{V}(m,t)+T_C(t)+\ 2cT_D(t)+c^2T_E(t)$$
$$=Mg\int_0^L H(x-ct)Sin\frac{m\pi x}{L}dx \qquad (3.6)$$

where

$$T_A(t)=\left(\frac{m\pi}{L}\right)^4\int_0^L Sin\frac{m\pi x}{L}dx \qquad (3.7)$$

$$T_B(t)=\frac{\partial^2}{\partial t^2}\left\{\frac{1}{L}\sum_{k=1}^{\infty}\frac{\mu}{V_k}\bar{V}(k,t)\frac{k^2\pi^2}{L}\int_0^L Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx\right\} \qquad (3.8)$$

$$T_C(t)=M\int_0^L H(x-ct)\frac{\partial^2 V(x,t)}{\partial t^2}Sin\frac{m\pi x}{L}dx \qquad (3.9)$$

$$T_D(t)=M\int_0^L H(x-ct)\frac{\partial^2 V(x,t)}{\partial x\partial t}Sin\frac{m\pi x}{L}dx \qquad (3.10)$$

$$T_E(t)=M\int_0^L H(x-ct)\frac{\partial^2 V(x,t)}{\partial x^2}Sin\frac{m\pi x}{L}dx \qquad (3.11)$$

In order to evaluate the integrals in (3.9), (3.10) and (3.11), use is made of the Fourier series representation of the Heaviside step function given as

$$H(x-ct)=\frac{1}{4}+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin\big((2n+1)\pi(x-ct)\big)}{2n+1},\quad 0<x<L \qquad (3.12)$$

Adopting (3.12), equations (3.9)...(3.11) simplify into

$$T_C(t)=\frac{M\mu}{V_k}\sum_{k=1}^{\infty}\bar{V}_{tt}(k,t)\left[\frac{1}{4}\int_0^L Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}.\right.$$
$$\int_0^L Sin(2n+1)\pi x Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}.$$
$$\left.\int_0^L Cos(2n+1)\pi x Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx\right] \qquad (3.13)$$

$$T_D(t)=\frac{M\mu}{V_k}\sum_{k=1}^{\infty}\bar{V}_t(k,t)\frac{k\pi}{L}\left[\frac{1}{4}\int_0^L Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}.\right.$$
$$\frac{k\pi}{L}\int_0^L Sin(2n+1)\pi x Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}.$$
$$\left.\frac{k\pi}{L}\int_0^L Cos(2n+1)\pi x Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx\right] \qquad (3.14)$$

$$T_E(t)=-\frac{M\mu}{V_k}\sum_{k=1}^{\infty}\bar{V}(k,t)\frac{k\pi}{L}\left[\frac{1}{4}\int_0^L Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}.\right.$$

$$\int_0^L Sin(2n+1)\pi x Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}.$$
$$\left.\int_0^L Cos(2n+1)\pi x Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx\right] \quad (3.15)$$

Substituting (3.7), (3.8), (3.13)...(3.15) into (3.6) with some simplifications and rearrangements, one obtains

$$\bar{V}_{tt}(m,t) + G_A\bar{V}(m,t) - r^0\sum_{k=1}^{\infty}G_B\bar{V}_{tt}(k,t) + \Gamma_0 L\left\{\sum_{k=1}^{\infty}\left[\left\{\frac{1}{4}G_D + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}.G_E\right.\right.\right.$$
$$\left. -\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}G_F\right\}\bar{V}_{tt}(k,t) + \left\{\frac{c}{2}G_G + \frac{2c}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}G_H - \right.$$
$$\left.\frac{2c}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}G_I\right\}\bar{V}_t(k,t) + \left\{\frac{c^2}{4}G_B + \frac{c^2}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}G_J\right.$$
$$\left.\left.\left. -\frac{c^2}{\pi}\sum_{n=0}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}G_L\right\}\bar{V}(k,t)\right]\right\} = \frac{PL}{\mu m\pi}\left[-Cosm\pi + Cos\frac{m\pi ct}{L}\right] \quad (3.16)$$

where

$$\Gamma_0 = \frac{M}{\mu L}, \quad P = Mg \quad (3.17)$$

and

$$G_A = \left(\frac{m\pi}{L}\right)^4 + \frac{K}{\mu}, \qquad G_B = \frac{k^2\pi^2}{V_k L^2}\int_0^L Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx$$

$$G_D = \frac{1}{V_k}\int_0^L Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx, \qquad G_E = \frac{1}{V_k}\int_0^L Sin(2n+1)\pi x Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx$$

$$G_F = \frac{1}{V_k}\int_0^L Cos(2n+1)\pi x Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx, \qquad G_G = \frac{k\pi}{V_k L}\int_0^L Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx$$

$$G_H = \frac{k\pi}{V_k L}\int_0^L Sin(2n+1)\pi x Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx, \qquad V_k = \int_0^L Sin\frac{k\pi x}{L}Sin\frac{k\pi x}{L}dx$$

$$G_I = \frac{k\pi}{V_k L}\int_0^L Cos(2n+1)\pi x Cos\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx$$

$$G_J = -\frac{k^2\pi^2}{V_k L^2}\int_0^L Sin(2n+1)\pi x(x) Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx$$

$$G_L = -\frac{k^2\pi^2}{V_k L^2}\int_0^L Cos(2n+1)\pi x Sin\frac{k\pi x}{L}Sin\frac{m\pi x}{L}dx \quad (3.18)$$

Solving integrals in (3.18), and substituting into equation (3.16) yields

$$\left(1 + \frac{R_0 m^2\pi^2}{L^2}\right)\bar{V}_{tt}(m,t) + \left(\frac{EI}{\mu}\left(\frac{m\pi}{L}\right)^4 + \frac{K}{\mu}\right)\bar{V}(m,t) + \Gamma_0 L\left\{\left(\frac{1}{4} + \frac{L^2}{2\pi^2}\sum_{n=0}^{\infty}(2n+1).\right.\right.$$

$$\left[\frac{(-1)^{2m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-4m^2}-\frac{Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2}\right]\frac{Cos(2n+1)\pi ct}{2n+1}\Bigg)\bar{V}_{tt}(m,t)$$

$$-\frac{2mc}{\pi L}\sum_{n=0}^{\infty}\left[2m\left\{\frac{(-1)^{2m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-4m^2}\right\}\right]\frac{Sin(2n+1)\pi ct}{2n+1}\bar{V}_t(m,t)-\frac{c^2m^2\pi^2}{L^2}.$$

$$\left(\frac{1}{4}+\frac{L^2}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\left[\frac{(-1)^{2m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-4m^2}-\frac{Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2}\right]\right)\right)\bar{V}(m,t)\Bigg\}$$

$$+\frac{\Gamma_0 L}{V_k}\left\{\frac{L^2}{2\pi^2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(2n+1)\frac{Cos(2n+1)\pi ct}{2n+1}\left[\frac{(-1)^{k+m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-(k+m)^2}\right.\right.$$

$$\left.-\frac{(-1)^{k-m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-(k-m)^2}\right]\bar{V}_{tt}(k,t)-\frac{2Lc}{2\pi^2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\left(\frac{k\pi}{L}\right)\frac{Sin(2n+1)\pi ct}{2n+1}\bar{V}_t(k,t).$$

$$\left[\frac{(k+m)\{(-1)^{k+m}Cos(2n+1)\pi L-1\}}{\left((2n+1)L\right)^2-(k+m)^2}-\frac{(k-m)\{(-1)^{k-m}Cos(2n+1)\pi L-1\}}{\left((2n+1)L\right)^2-(k-m)^2}\right]$$

$$+\frac{c^2}{\pi}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(2n+1)L^2}{2\pi}\left[\frac{(-1)^{k+m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-(k+m)^2}-\frac{(-1)^{k-m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-(k-m)^2}\right].$$

$$\frac{Cos(2n+1)\pi ct}{2n+1}\left(-\frac{k^2\pi^2}{L^2}\right)\bar{V}(k,t)\Bigg\}=\frac{PL}{\mu m\pi}\left[-(-1)^m+Cos\frac{m\pi ct}{L}\right] \qquad (3.19)$$

Equation (3.19) is now the fundamental equation of our problem when the uniform Rayleigh beam has simple supports at all edges. In what follows, we shall discuss two cases of the equation.

3.1 Simply Supported Uniform Rayleigh Beam Traversed By Moving Distributed Force

An approximate model of the system when the inertia effect of moving distributed mass M is neglected is called moving distributed force problem and it is obtained when Γ_0 is set to zero in (3.19). Thus, the moving distributed force problem associated with the system is given as

$$\left(1+\frac{r^0m^2\pi^2}{L^2}\right)\bar{V}_{tt}(m,t)+\left(\frac{EI}{\mu}\left(\frac{m\pi}{L}\right)^4+\frac{K}{\mu}\right)\bar{V}(m,t)=\frac{PL}{\mu m\pi}\left[-(-1)^m+Cos\frac{m\pi ct}{L}\right] \qquad (3.20)$$

which when rearranged gives

$$\bar{V}_{tt}(m,t)+\gamma_{ff}^2\bar{V}(m,t)=P_f\left[-(-1)^m+Cos\frac{m\pi ct}{L}\right] \qquad (3.21)$$

where

$$\gamma_{ff}^2 = \frac{\left[\frac{EI}{\mu}\left(\frac{m\pi}{L}\right)^4 + \frac{K}{\mu}\right]}{1 + r^0\left(\frac{m^2\pi^2}{L^2}\right)}, \qquad P_f = \frac{PL}{\mu m\pi\left[1 + r^0\left(\frac{m^2\pi^2}{L^2}\right)\right]} \tag{3.22}$$

Solving equation (3.21) using the method of Laplace transforms and Convolution theory in conjunction with the initial conditions (2.14), one obtains

$$\bar{V}(m,t) = P_f\left[\frac{Cos\alpha_c t - Cos\gamma_{ff}t}{\gamma_{ff}^2 - \alpha_c^2} + \frac{(-1)^m Cos\gamma_{ff}t}{\gamma_{ff}} - \frac{(-1)^m}{\gamma_{ff}}\right] \tag{3.23}$$

where

$$\alpha_c = \frac{m\pi c}{L} \tag{3.24}$$

Substituting (3.23) into (3.2), we obtain

$$V(x,t) = \frac{2}{L}\sum_{m=1}^{\infty} P_f\left[\frac{Cos\alpha_c t - Cos\gamma_{ff}t}{\gamma_{ff}^2 - \alpha_c^2} + \frac{(-1)^m Cos\gamma_{ff}t}{\gamma_{ff}} - \frac{(-1)^m}{\gamma_{ff}}\right] Sin\frac{m\pi x}{L} \tag{3.25}$$

as the transverse displacement response to a distributed force moving at constant velocity of a uniform simply supported Rayleigh beam resting on elastic foundation.

3.2 Simply Supported Uniform Rayleigh Beam Traversed By Moving Distributed Mass

This section seeks the solution of the entire equation (3.19) subjected to the initial conditions (2.14) when all the inertia terms are considered. In this case $\Gamma_0 \neq 0$. This is termed the moving mass problem. Evidently, an exact solution to this problem is impossible. Thus, a technique which is based on a modification of Struble's technique discussed in [14] is resorted to. To this end, equation (3.19) is rearranged to take the form

$$\bar{V}_{tt}(m,t) - \frac{2mc\Gamma_0 LR_{II}(m,n,t)}{\pi L\left(1 + \Gamma_0 LR_I(m,n,t)\right)}\bar{V}_t(m,t) - \frac{\left(\frac{c^2m^2\pi^2}{L^2}\Gamma_0 LR_I(m,n,t) - \gamma_{ff}^2\right)}{\left(1 + \Gamma_0 LR_I(m,n,t)\right)}\bar{V}(m,t)$$

$$+\frac{\Gamma_0 L}{V_k\left(1 + \Gamma_0 LR_I(m,n,t)\right)}\left\{\frac{L^2}{2\pi^2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(2n+1).\left[\frac{(-1)^{k+m}Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2 - (k+m)^2} - \right.\right.$$

$$\left.\frac{(-1)^{k-m}Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2 - (k-m)^2}\right].\frac{Cos(2n+1)\pi ct}{2n+1}\bar{V}_{tt}(k,t) - \frac{c}{\pi}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{Sin(2n+1)\pi ct}{2n+1}.$$

$$k\left[\frac{(k+m)\{(-1)^{k+m}Cos(2n+1)\pi L - 1\}}{\left((2n+1)L\right)^2 - (k+m)^2}\frac{(k-m)\{(-1)^{k-m}Cos(2n+1)\pi L - 1\}}{\left((2n+1)L\right)^2 - (k-m)^2}\right]\bar{V}_t(k,t)$$

$$-\frac{c^2}{2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(2n+1)k^2\left[\frac{(-1)^{k+m}Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2 - (k+m)^2} - \frac{(-1)^{k-m}Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2 - (k-m)^2}\right].$$

$$\frac{Cos(2n+1)\pi ct}{2n+1}\bar{V}(k,t)\} = \frac{PL}{\mu m\pi\left(1+\Gamma_0 LR_I(m,n,t)\right)}\left[-(-1)^m + Cos\frac{m\pi ct}{L}\right] \tag{3.26}$$

where

$$R_I(m,n,t) = \frac{1}{4} + \frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1).\left[\frac{(-1)^{2m}Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2 - 4m^2} - \frac{Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2}\right]. \frac{Cos(2n+1)\pi ct}{2n+1} \tag{3.27}$$

$$R_{II}(m,n,t) = \sum_{n=0}^{\infty}\left[2m\left\{\frac{(-1)^{2m}Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2 - 4m^2}\right\}\right]\frac{Sin(2n+1)\pi ct}{2n+1} \tag{3.28}$$

First, we shall consider the homogeneous part of (3.26) and obtain a modified frequency corresponding to the frequency of the free system due to the presence of the moving distributed mass. An equivalent free system operator defined by the modified frequency then replaces equation (3.26). Thus, the right hand of equation (3.26) is set to zero and a parameter $\Gamma_1 < 1$ is considered for any arbitrary mass ratio Γ_0 , defined as

$$\Gamma_1 = \frac{\Gamma_0}{1+\Gamma_0} \tag{3.29}$$

Evidently

$$\Gamma_0 = \Gamma_1 + o(\Gamma_1^2) \tag{3.30}$$

which implies

$$\frac{1}{\left(1+\Gamma_0 LR_I(m,n,t)\right)} = \left(1-\Gamma_1 LR_I(m,n,t)\right) + O(\Gamma_1^2) \tag{3.31}$$

and

$$|1-\Gamma_1 LR_I(m,n,t)| < 1 \tag{3.32}$$

Setting $\Gamma_1 = 0$, a situation corresponding to the case in which the inertia effect of the mass of the system is regarded as negligible is obtained, then the solution of (3.26) can be written as

$$\bar{V}(m,t) = A_0 Cos\left[\gamma_{ff}t - \psi_m\right] \tag{3.32}$$

where A_0 and ψ_m are constants.

Since $\Gamma_1 < 1$, Struble's technique requires that the asymptotic solution of the homogeneous part of equation (3.26) be of the form [15]

$$\bar{V}(m,t) = \Delta(m,t)Cos\left[\gamma_{ff}t - \phi(m,t)\right] + \Gamma_1\bar{V}(m,t) + o(\Gamma_1^2) \tag{3.33}$$

where $\Delta(m,t)$ and $\phi(m,t)$ are slowly varying functions of time.

Substituting equations (3.33) and its derivatives into the homogeneous part of equation (3.26) while taking into account (3.30) and retaining terms to $o(\Gamma_1)$, one obtains

$$-2\dot{\Delta}(m,t)\gamma_{ff}Sin\left[\gamma_{ff}t - \phi(m,t)\right] + 2\Delta(m,t)\dot{\phi}(m,t)\gamma_{ff}Cos\left[\gamma_{ff}t - \phi(m,t)\right]$$

$$+\frac{2mc\Gamma_1 L}{\pi L}\sum_{n=0}^{\infty}\left[2m\left\{\frac{(-1)^{2m}Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2 - 4m^2}\right\}\right]\frac{Sin(2n+1)\pi ct}{2n+1}\gamma_{ff}Sin\left[\gamma_{ff}t - \phi(m,t)\right].$$

$$\Delta(m,t) - \frac{c^2m^2\pi^2\Gamma_1 L}{L^2}\left(\frac{1}{4} + \frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1)\frac{Cos(2n+1)\pi ct}{2n+1}\Delta(m,t)Cos\left[\gamma_{ff}t - \phi(m,t)\right].\right.$$

$$\left.\left[\frac{(-1)^{2m}Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2 - 4m^2} - \frac{Cos(2n+1)\pi L - 1}{\left((2n+1)L\right)^2}\right]\right) - \gamma_{ff}^2\Gamma_1 L\left(\frac{1}{4} + \frac{L}{\pi^2}\sum_{n=0}^{\infty}(2n+1).\right.$$

$$\left[\frac{(-1)^{2m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-4m^2}-\frac{Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2}\right]\frac{Cos(2n+1)\pi ct}{2n+1}\Bigg).$$

$$\Delta(m,t)Cos\left[\gamma_{ff}t-\phi(m,t)\right]+\frac{\Gamma_1 L}{V_k}\Bigg\{\frac{L^2}{2\pi^2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(2n+1).\frac{Cos(2n+1)\pi ct}{2n+1}\Delta_{11}(k,t).$$

$$\left[\frac{(-1)^{k+m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-(k+m)^2}-\frac{(-1)^{k-m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-(k-m)^2}\right]$$

$$-\frac{c}{\pi}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}k\frac{Sin(2n+1)\pi ct}{2n+1}\left[\frac{(k+m)\{(-1)^{k+m}Cos(2n+1)\pi L-1\}}{\left((2n+1)L\right)^2-(k+m)^2}\right.$$

$$\left.-\frac{(k-m)\{(-1)^{k-m}Cos(2n+1)\pi L-1\}}{\left((2n+1)L\right)^2-(k-m)^2}\right]\Delta_{12}(k,t)+\frac{c^2}{2}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(2n+1)k^2\,\Delta_{13}(k,t).$$

$$\left[\frac{(-1)^{k+m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-(k+m)^2}-\frac{(-1)^{k-m}Cos(2n+1)\pi L-1}{\left((2n+1)L\right)^2-(k-m)^2}\right]\frac{Cos(2n+1)\pi ct}{2n+1}\Bigg\} \quad (3.34)$$

where

$$\Delta_{11}(k,t)=-2\dot{\Delta}(k,t)\gamma_{ff}Sin\left[\gamma_{ff}t-\phi(k,t)\right]-2\Delta(k,t)\dot{\phi}(k,t)\gamma_{ff}.$$

$$Cos\left[\gamma_{ff}t-\phi(k,t)\right]-\Delta(k,t)\gamma_{ff}^2Cos\left[\gamma_{ff}t-\phi(k,t)\right]$$

$$\Delta_{12}(k,t)=\dot{\Delta}(k,t)Cos\left[\gamma_{ff}t-\phi(k,t)\right]+\Delta(k,t)\dot{\phi}(k,t).$$

$$Sin\left[\gamma_{ff}t-\phi(k,t)\right]-\Delta(k,t)\gamma_{ff}Sin\left[\gamma_{ff}t-\phi(k,t)\right]$$

$$\Delta_{13}(k,t)=\Delta(k,t)Cos\left[\gamma_{ff}t-\phi(k,t)\right] \quad (3.35)$$

The variational equations are obtained by equating the coefficients of $Sin\left[\gamma_{ff}t-\phi(m,t)\right]$ and $Cos\left[\gamma_{ff}t-\phi(m,t)\right]$ on both sides of equation (3.34). Hence, noting the following trigonometric identities,

$$\frac{Cos(2n+1)\pi ct}{2n+1}Sin\left[\gamma_{ff}t-\phi(m,t)\right]=\frac{1}{2}Sin\left[\frac{(2n+1)\pi ct}{2n+1}+\gamma_{ff}t-\phi(m,t)\right]$$

$$-\frac{1}{2}Sin\left[\frac{(2n+1)\pi ct}{2n+1}-\gamma_{ff}t+\phi(m,t)\right] \quad (3.36)$$

$$\frac{Sin(2n+1)\pi ct}{2n+1}Sin\left[\gamma_{ff}t-\phi(m,t)\right]=\frac{1}{2}Cos\left[\frac{(2n+1)\pi ct}{2n+1}-\gamma_{ff}t+\phi(m,t)\right]$$

$$-\frac{1}{2}Cos\left[\frac{(2n+1)\pi ct}{2n+1}+\gamma_{ff}t-\phi(m,t)\right] \quad (3.37)$$

$$\frac{Cos(2n+1)\pi ct}{2n+1}Cos\left[\gamma_{ff}t-\phi(m,t)\right]=\frac{1}{2}Cos\left[\frac{(2n+1)\pi ct}{2n+1}+\gamma_{ff}t-\phi(m,t)\right]$$
$$+\frac{1}{2}Cos\left[\frac{(2n+1)\pi ct}{2n+1}-\gamma_{ff}t+\phi(m,t)\right] \qquad (3.38)$$

$$\frac{Sin(2n+1)\pi ct}{2n+1}Cos\left[\gamma_{ff}t-\phi(m,t)\right]=\frac{1}{2}Sin\left[\frac{(2n+1)\pi ct}{2n+1}+\gamma_{ff}t-\phi(m,t)\right]$$
$$+\frac{1}{2}Sin\left[\frac{(2n+1)\pi ct}{2n+1}-\gamma_{ff}t+\phi(m,t)\right] \qquad (3.39)$$

and neglecting terms that do not contribute to the variational equation, equation (3.34) reduces to

$$-2\dot{\Delta}(m,t)\gamma_{ff}Sin\left[\gamma_{ff}t-\phi(m,t)\right]+2\Delta(m,t)\dot{\phi}(m,t)\gamma_{ff}Cos\left[\gamma_{ff}t-\phi(m,t)\right]$$

$$-\frac{c^2m^2\pi^2\Gamma_1L}{L^2}.\frac{1}{4}\Delta(m,t)Cos\left[\gamma_{ff}t-\phi(m,t)\right]-\gamma_{ff}^2\Gamma_1L.\frac{1}{4}\Delta(m,t)Cos\left[\gamma_{ff}t-\phi(m,t)\right]=0$$

(3.40)

Then, the variational equations are respectively

$$-2\dot{\Delta}(m,t)\gamma_{ff}=0 \qquad (3.41)$$

and

$$2\Delta(m,t)\dot{\phi}(m,t)\gamma_{ff}-\frac{c^2m^2\pi^2\Gamma_1L}{L^2}.\frac{1}{4}\Delta(m,t)-\gamma_{ff}^2\Gamma_1L.\frac{1}{4}\Delta(m,t)=0 \qquad (3.42)$$

Solving equations (3.41) and (3.42) respectively, one obtains

$$\Delta(m,t)=\Delta_m \qquad (3.43)$$

and

$$\phi(m,t)=\left(\frac{c^2m^2\pi^2}{L^2}+\gamma_{ff}^2\right)\frac{\Gamma_1L}{8\gamma_{ff}}t+\phi_m \qquad (3.44)$$

where Δ_m and ϕ_m are constants.
Therefore when the effects of the mass of the particle is considered, the first approximation to the homogeneous system is

$$\bar{V}(m,t)=\Delta_m Cos[\Omega_{mm}t-\phi_m] \qquad (3.45)$$

where

$$\Omega_{mm}=\frac{8\gamma_{ff}^3L^2-\Gamma_1L\{\gamma_{ff}^2L^2+c^2m^2\pi^2\}}{8\gamma_{ff}^2L^2} \qquad (3.46)$$

is called the modified natural frequency corresponding to the frequency of the free system due to the presence of the distributed moving mass.
Thus to solve the non-homogeneous equation (3.26), the differential operator which acts on $\bar{V}(m,t)$ and $\bar{V}(k,t)$ is replaced by the modified frequency Ω_{mm} , i.e

$$\bar{V}_{tt}(m,t)+\Omega_{mm}^2\bar{V}(m,t)=\frac{PL}{\mu m\pi}\left[-(-1)^m+Cos\frac{m\pi ct}{L}\right] \qquad (3.47)$$

Solving equation (3.47) by methods of Laplace transforms and Convolution theory in conjunction with the initial condition, one obtains expression for $\bar{V}(m,t)$. Thus, in view of (3.2)

$$V(x,t)=\sum_{m=1}^{\infty}\frac{2\Gamma_1 Lg}{m\pi}\left[\frac{Cos\alpha_c t-Cos\Omega_{mm}t}{\Omega_{mm}^2-\alpha_c^2}+\frac{(-1)^m Cos\Omega_{mm}t}{\Omega_{\mathrm{mm}}}-\frac{(-1)^m}{\Omega_{mm}}\right]Sin\frac{m\pi x}{L} \quad (3.48)$$

Equation (3.48) represents the transverse displacement response to a distributed mass moving with constant velocity of a simply supported uniform Rayleigh beam resting on an elastic foundation.

4.0 Discussion of the Analytical Solutions

At this point, it is important to establish conditions under which resonance occurs for an undamped system such as this. Resonance takes place when the transverse displacement of the vibrating structure increase without bound. In actual practice, when this happens, the structure collapses as the intensive vibrations cause cracks or permanent deformation in the vibrating structures.

Equation (3.25) clearly shows that the simply supported uniform Rayleigh beam resting on elastic foundation and traversed by moving distributed force reaches a state of resonance whenever

$$\gamma_{ff}=\frac{m\pi c}{L} \quad (4.1)$$

while equation (3.48) indicates that the same beam under the action of a moving distributed mass experiences resonance effect when,

$$\Omega_{mm}=\frac{m\pi c}{L} \quad (4.2)$$

Evidently,

$$\Omega_{mm}=\gamma_{ff}\left[1-\frac{\Gamma_1 L}{8\gamma_{ff}}\left\{1+\frac{c^2m^2\pi^2}{\gamma_{ff}^2L^2}\right\}\right]=\frac{m\pi c}{L} \quad (4.3)$$

Equations (4.1) and (4.3) show that for the same natural frequency, the critical speed for the system consisting of a simply supported uniform Rayleigh beam resting on an elastic foundation and traversed by a distributed force moving with a uniform velocity is greater than that of the distributed moving mass. Thus, resonance is reached earlier in the distributed moving mass system than in the distributed moving force system.

5.0 Numerical Results and Discussion

In order to illustrate the foregoing analysis, the simply supported uniform Rayleigh beam is taken to be of length L=12.192m, the load velocity, c=8.128ms^{-1} and E=2109$\times10^9$kg/m. The values of the rotatory inertia correction factor r^0 are varied between 0.5 and 9.5, while the values of the foundation moduli constant K are varied between 0 and 4000000Nm^2.The flexural vibrations of the simply supported uniform Rayleigh beam are calculated and graphs are plotted for beam response against time for values of rotatory inertia correction factor r^0 and foundation moduli constant K.

Fig 5.1, displays the displacement response of a finite uniform simply supported Rayleigh beam to moving distributed force for various values of foundation moduli K and fixed rotatory inertia correction factor r^0=7.5. The graph shows that the response amplitude decreases as the value of K increases. In Fig 5.2, the displacement response of a finite uniform simply supported Rayleigh to moving distributed force for various values of rotatory inertia correction factor r^0 and fixed foundation modulus K =40000 is displayed. The graph shows that the response amplitude decreases as the value of r^0 increases. Fig 5.3, depicts the deflection profile of a finite uniform simply supported Rayleigh beam to moving distributed mass for various values of foundation moduli K and fixed rotatory inertia correction factor r^0=7.5. The graph shows that the response amplitude decreases as the value of K increases. Also, in Fig 5.4, the deflection profile of a finite uniform simply supported Rayleigh beam to moving distributed mass for various values of rotatory inertia correction factor r^0 and fixed foundation modulus K =40000. The graph shows that the response amplitude decreases as the value of r^0 increases. Fig 5.5, shows comparison of response to moving distributed force and moving distributed mass for the simply supported finite uniform Rayleigh beam for fixed foundation modulus K =400000 and fixed rotatory inertia correction factor r^0=7.5. Clearly the response amplitude of the moving distributed mass system is higher than that of the moving distributed force system. This clearly confirms that the moving distributed force solution is not always an upper bound to the solution of the moving distributed mass system. This interesting result has been reported in [ ] for the cases when the travelling load is modelled as concentrated load.

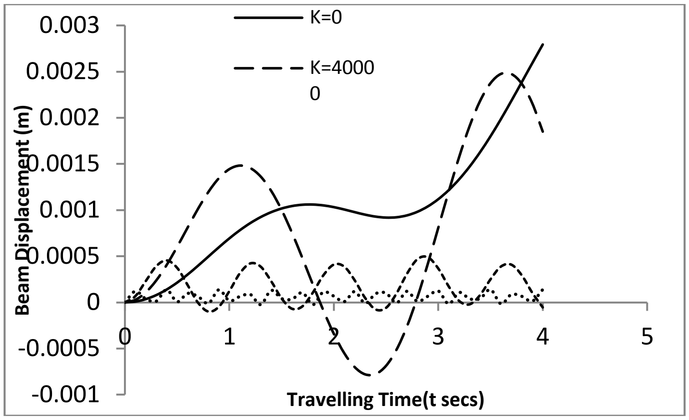

Fig 5.1: Displacement response of a uniform simply supported Rayleigh beam to distributed forces for various values of foundation moduli K

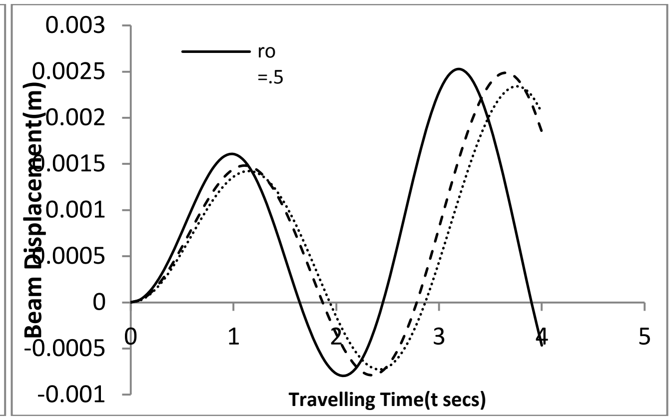

Fig 5.2: Displacement response of a uniform simply supported Rayleigh beam to distributed forces for various

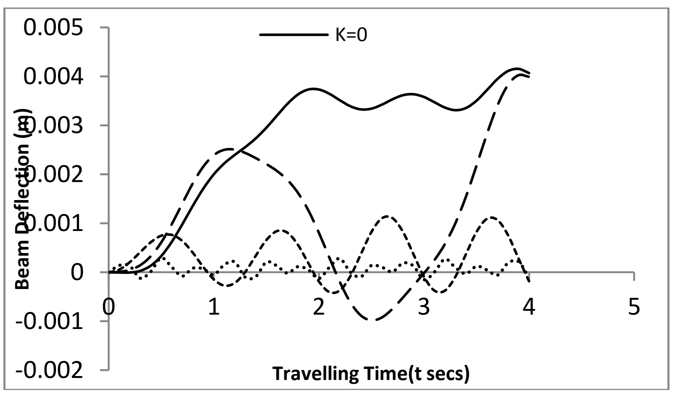

Fig 5.3: Deflection profile of a uniform simply supported Rayleigh beam to distributed masses for various values of foundation moduli K.

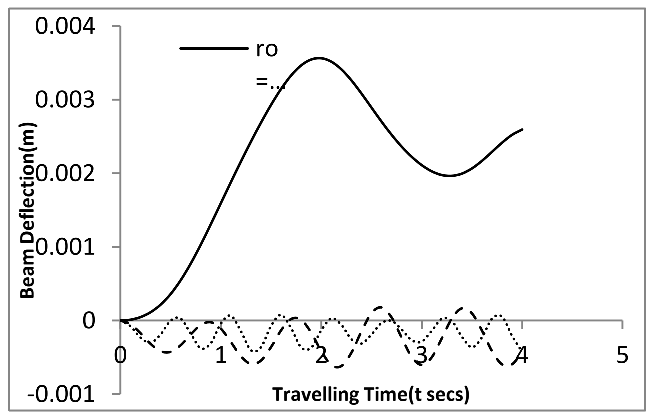

Fig 5.4: Deflection profile of a uniform simply supported Rayleigh beam to distributed masses for various values of rotatory inertia correction factor r^0

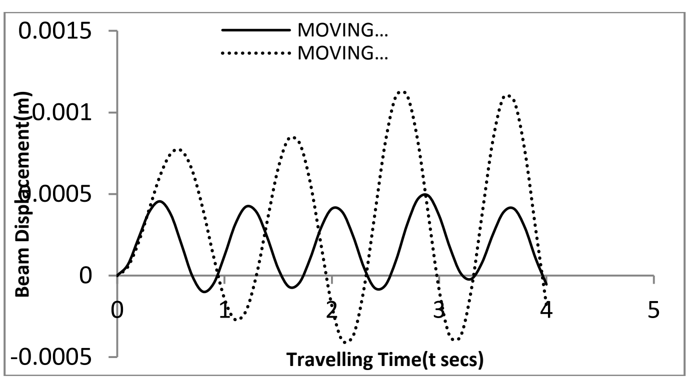

Fig 5.5: Comparison of the displacement response of moving distributed force and moving distributed mass cases of a uniform simply supported Rayleigh beam for Ro=7.5 and K=400000

6.0 Concluding Remarks

In this paper, an analysis of the flexural motions of a finite simply supported uniform Rayleigh beam subjected to travelling distributed loads has been presented. Both gravity and inertia effects of the distributed loads were taken into consideration. The fourth order partial differential equation governing the motion of the beam was solved using the generalized integral transformation technique and the modified asymptotic method of Struble. The deflection of the beam having simple supports at both ends was calculated and shown graphically for various values of foundation moduli and rotatory inertia correction factor. It was found that as the value of foundation moduli is increased, the displacement response of the beam decreases. Also, as the rotatory inertia correction factor increases, results show that the deflection of the beam model decreases. It is deduced that for the same natural frequency, the critical speed for the system consisting of a simply supported uniform Rayleigh beam resting on an elastic foundation and traversed by a distributed force moving with a uniform velocity is greater than that of the distributed moving mass problem. Thus, resonance is reached earlier in the distributed moving mass system than in the distributed moving force system. It is further seen that the response amplitude of the moving distributed mass system is higher than that of the moving distributed force system for fixed values of rotatory inertia correction factor and foundation moduli. Thus for the simply supported moving distributed load problem, it is established that the moving distributed force solution is not an upper bound for an accurate solution of the moving distributed mass problem.

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