Journal of the Nigerian Association of Mathematical Physics
Volume 26, (*March, 2014*), pp 109 – 124

Flexural Vibrations Under Moving Masses of Rectangular Plates With General Boundary Conditions and Resting on Variable Bi-Parametric Foundation

Awodola T. O. and Oni S. T.

Department of Mathematical Sciences
Federal University of Technology, Akure, Nigeria.

Abstract

In this paper, the dynamic analysis of rectangular plate, with general classical boundary conditions, carrying moving masses and resting on a bi-parametric (Pasternak) elastic foundation with stiffness variation is considered. The governing equation is a fourth order partial differential equation with variable and singular coefficients, in order to solve the governing equation, a technique based on separation of variables is used to reduce the equation to a sequence of second order ordinary differential equations. The Struble's technique and the integral transformations are employed for the solutions of the second order ordinary differential equations. The results are then presented in plotted curves. The results show that as the value of the rotatory inertia correction factor R_o increases, the response amplitudes of the plate decrease and that, for fixed value of R_o, the displacements of the plate decrease as the foundation modulus F_o increases for the variants of the classical boundary conditions considered. The results also show that for fixed R_o and F_o, the transverse deflections of the rectangular plates under the actions of moving masses are higher than those when only the force effects of the moving load are considered. For the rectangular plate, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem for all variants of classical boundary conditions, that is, resonance is reached earlier in moving mass problem than in moving force problem, this implies that the moving force solution is not a save approximation to the moving mass problem and hence it is highly risky to rely on the moving force solution as an approximate solution to the moving mass problem.

Keywords: Pasternak Foundation, Critical speed, Rotatory Inertia, Resonance, Moving Force, Moving Mass, Foundation modulus.

1.0 Introduction

The plate, or flexural member, is frequently encountered in structures and machines. Elastic structures ranging from bridges and roads to space vehicles are constantly acted upon by moving loads and hence, the problem of analyzing the dynamic response of elastic structures under the action of moving masses continues to motivate a variety of investigations [1-6].

Generally, the dynamical problems of structures under moving loads and resting on a foundation are complex. The complexity increases if the foundation stiffness varies along the span of the structure. Aside the problem of singularity brought in by the inclusion of the inertia effects of the moving load, the coefficients of the governing fourth order partial differential equation are no longer constant but variable. Earlier researchers into beam member on variable elastic foundation include Franklin and Scott [7] who presented a closed-form solution to a linear variation of the foundation modulus using contour-integrals. Closely following this, Lentini [8] presented a finite difference method to solve the problem where the foundation stiffness varies along x as a power of x. These works, though useful, considered the loads acting on the beams to be static (not moving). Recently, Oni and Awodola [9] extended the works of these previous authors to investigate the dynamic response to moving concentrated masses of uniform Rayleigh beams resting on variable Winkler elastic foundation. Oni and Awodola [10] again considered the dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation. The technique was based on the generalized Galerkin's method and integral transformations.

Corresponding author: ***Awodola T. O.,*** E-mail: -, Tel.: +2348030796073

The foundation model based on Winkler's approximation model is very common in literature, whereas, in such an important Engineering problem as the vibration of plates resting on elastic foundation, a more accurate Two-Parameter (Pasternak) foundation model which in addition to subgrade modulus incorporates the shear effect of the foundation should be used rather than the Winkler's approximation model. Eisenberger and Clastornik [11] presented two methods for the solution of beams on variable two-parameter elastic foundation. Also, Gbadeyan and Oni [12] studied the dynamic analysis of an elastic plate continuously supported by an elastic Pasternak foundation traversed by an arbitrary number of concentrated masses. In their work, they assumed that both the foundation modulus and the shear modulus are constants.

Several researchers have also made tremendous efforts in the study of dynamics of structures under moving loads [13 - 21]. Aside the problem arising from the inclusion of the inertia terms in moving mass problems, difficulties often arise from the type of specified end-conditions. There are four classical boundary conditions that are commonly of practical interest to an applied Mathematician or an Engineer. These are Pinned end conditions (Simply supported end conditions), Fixed / Clamped end conditions, Free end conditions and Sliding end conditions [22].

In most of the investigations in literature on vibration of rectangular plate under moving loads and resting on elastic foundations, work has been restricted to cases when the elastic foundations are regarded as being constant. The more complicated case, when the elastic foundation varies along the span of the structure has been neglected, where this is considered, work has been restricted to the simplest forms of the problem when the structure is simply supported [23] or when the foundation model is based on the simple and common Winkler's approximation model [24]. This paper is therefore concerned with the problem of assessing the dynamic response to moving concentrated masses of rectangular plates with **general classical boundary conditions** and resting on **variable Pasternak elastic foundations**.

2.0 Governing Equation

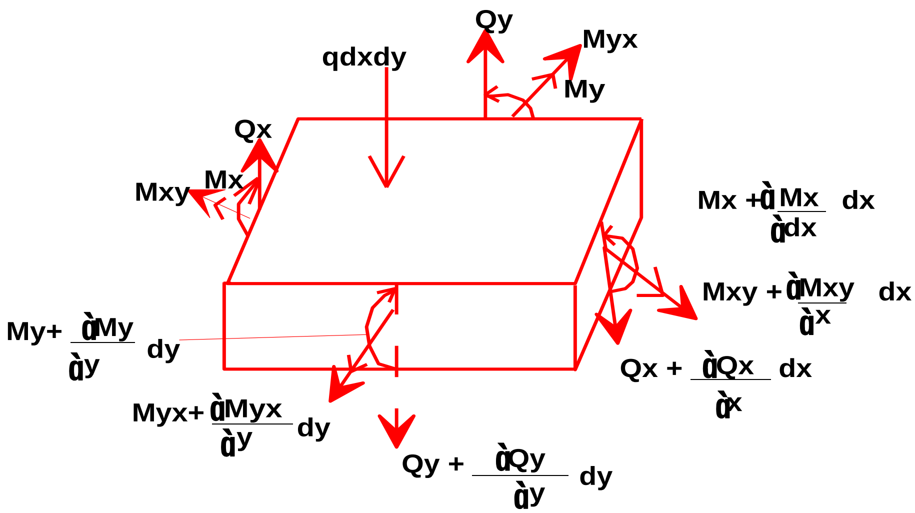

Figure 2.1.

Figure 2.1 is a differential element of a plate showing the various shearing forces, bending and twisting moments and external loads acting. The bending moment per unit length M_x, M_y arise from the distribution of normal stresses while the twisting moment per unit length M_{xy} and M_{yx} (shown as double cross-vectors) arise from shearing stresses. The shear forces per unit length Q_x and Q_y also arise from shearing stresses.

Consider a rectangular plate carrying an arbitrary number (say N) of concentrated masses M_i moving with constant velocities c_i, i = 1, 2, 3,…,N along a straight line parallel to the x-axis issuing from point y = s on the y-axis. The equation governing the dynamic transverse displacement V(x,y,t) of the rectangular plate when it is resting on a variable Pasternak foundation and traversed by several moving concentrated masses is the fourth order partial differential equation given by [23]

$$D\nabla^4 V(x,y,t) + \mu \frac{\partial^2 V(x,y,t)}{\partial t^2} = \mu R_0 \left[\frac{\partial^4}{\partial t^2 \partial x^2} + \frac{\partial^4}{\partial t^2 \partial y^2}\right] V(x,y,t)$$
$$- F_0\left[4x - 3x^2 + x^3\right]V(x,y,t) + G_0[-13 + 12x - 3x^2]\frac{\partial}{\partial x}V(x,y,t)$$
$$+ G_0[12 - 13x + 6x^2 - x^3]\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]V(x,y,t) + \sum_{i=1}^{N}\left[M_i g \delta(x - c_i t)\delta(y - s)\right. \tag{1}$$
$$\left. - M_i\left(\frac{\partial^2}{\partial t^2} + 2c_i \frac{\partial^2}{\partial t \partial x} + c_i^2 \frac{\partial^2}{\partial x^2}\right)V(x,y,t)\delta(x - c_i t)\delta(y - s)\right]$$

where D is the bending rigidity of the plate, μ is mass per unit area of the plate, x is the position co-ordinate in x – direction, y is position co-ordinate in y – direction, t is the time, R_0 is the rotatory inertia correction factor, ∇^2 is the two-dimensional Laplacian operator, F_0 is the foundation modulus, G_0 is the shear modulus, g is the acceleration due to gravity and δ (.) is the Dirac-Delta function.

At this juncture, the boundary condition is arbitrary and the initial condition, without any loss of generality, is taken as

$$V(x,y,t) = 0 = \frac{\partial V(x,y,t)}{\partial t} \tag{2}$$

3.0 Analytical Approximate Solution

In this section, we seek to obtain the analytical solution to the problem of the dynamic response of a rectangular plate resting on Variable Pasternak elastic foundation and subjected to arbitrary support conditions. The method of analysis involves expressing the Dirac – Delta function as a Fourier cosine series. A technique [13] based on separation of variables is used to tackle the fourth order partial differential equation governing the motion of the plate and reduce it to a set of coupled second order ordinary differential equations. Then, the modified asymptotic method of Struble in conjunction with the techniques of integral transformation and convolution theory are then employed to obtain the closed form solution of the resulting second order ordinary differential equations.

In order to solve equation (1), in the first instance, the deflection is written in the form [13]

$$V(x,y,t) = \sum_{n=1}^{\infty} \phi_n(x,y) T_n(t) \tag{3}$$

where ϕ_n are the known eigenfunctions of the plate with the same boundary conditions. The ϕ_n have the form of

$$\nabla^4 \phi_n - \omega_n^4 \phi_n = 0 \tag{4}$$

where $$\omega_n^4 = \frac{\Omega_n^2 \mu}{D} \tag{5}$$

Ω_n, n = 1, 2, 3, … , are the natural frequencies of the dynamical system and $T_n(t)$ are amplitude functions which have to be calculated.
In order to solve the equation (1), it is rewritten as

$$\frac{D}{\mu}\nabla^4 V(x,y,t) + \frac{\partial^2 V(x,y,t)}{\partial t^2} = R_0\left[\frac{\partial^4}{\partial t^2 \partial x^2} + \frac{\partial^4}{\partial t^2 \partial y^2}\right]V(x,y,t)$$
$$- \frac{F_0}{\mu}\left[4x - 3x^2 + x^3\right]V(x,y,t) + \frac{G_0}{\mu}[-13 + 12x - 3x^2]\frac{\partial}{\partial x}V(x,y,t) \tag{6}$$
$$+ \frac{G_0}{\mu}[12 - 13x + 6x^2 - x^3]\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]V(x,y,t) + \sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x - c_i t)\delta(y - s)\right.$$
$$\left. - \frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2} + 2c_i \frac{\partial^2}{\partial t \partial x} + c_i^2 \frac{\partial^2}{\partial x^2}\right)V(x,y,t)\delta(x - c_i t)\delta(y - s)\right]$$

The right hand side of equation (6) is written in the form of a series to have

$$R_0\left[\frac{\partial^4}{\partial t^2\partial x^2}+\frac{\partial^4}{\partial t^2\partial y^2}\right]V(x,y,t)-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]V(x,y,t)+\frac{G_0}{\mu}\left[\left(-13+12x-3x^2\right)\frac{\partial}{\partial x}V(x,y,t)\right.$$
$$\left.+\left(12-13x+6x^2-x^3\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V(x,y,t)\right]+\sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x-c_it)\delta(y-s)\right.$$
$$\left.-\frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2}+2c_i\frac{\partial^2}{\partial t\partial x}+c_i^2\frac{\partial^2}{\partial x^2}\right)V(x,y,t)\delta(x-c_it)\delta(y-s)\right]=\sum_{n=1}^{\infty}\phi_n(x,y)B_n(t) \tag{7}$$

Substituting equation (3) into equation (7) we have

$$\sum_{n=1}^{\infty}\Big\{R_0\left[\phi_{n,xx}(x,y)T_{n,tt}(t)+\phi_{n,yy}(x,y)T_{n,tt}(t)\right]-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_n(x,y)T_n(t)$$
$$+\frac{G_0}{\mu}\left[\left(-13+12x-3x^2\right)\phi_{n,x}(x,y)T_n(t)+\left(12-13x+6x^2-x^3\right)\left(\phi_{n,xx}(x,y)T_n(t)+\phi_{n,yy}(x,y)T_n(t)\right)\right]$$
$$+\sum_{i=1}^{N}\Big[\frac{M_i g}{\mu}\delta(x-c_it)\delta(y-s)-\frac{M_i}{\mu}\Big(\phi_n(x,y)T_{n,tt}(t)+2c_i\phi_{n,x}(x,y)T_{n,t}(t)$$
$$+c_i^2\phi_{n,xx}(x,y)T_n(t)\Big)\delta(x-c_it)\delta(y-s)\Big]\Big\}=\sum_{n=1}^{\infty}\phi_n(x,y)B_n(t) \tag{8}$$

where

$$\phi_{n,x}(x,y)\ implies\ \frac{\partial\phi_n(x,y)}{\partial x},\quad \phi_{n,xx}(x,y)\ implies\ \frac{\partial^2\phi_n(x,y)}{\partial x^2},$$
$$\phi_{n,y}(x,y)\ implies\ \frac{\partial\phi_n(x,y)}{\partial y},\quad \phi_{n,yy}(x,y)\ implies\ \frac{\partial^2\phi_n(x,y)}{\partial y^2}, \tag{9}$$
$$T_{n,t}(t)\ implies\ \frac{dT_n(t)}{dt}\ \ and\ \ T_{n,tt}(t)\ implies\ \frac{d^2T_n(t)}{dt^2}$$

Multiplying both sides of equation (8) by $\phi_p(x,y)$ and integrating on area A of the plate, we have

$$\sum_{n=1}^{\infty}\int_A\Big\{R_0\left[\phi_{n,xx}(x,y)\phi_p(x,y)T_{n,tt}(t)+\phi_{n,yy}(x,y)\phi_p(x,y)T_{n,tt}(t)\right]$$
$$-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_n(x,y)\phi_p(x,y)T_n(t)+\frac{G_0}{\mu}\Big[\left(-13+12x-3x^2\right)\phi_{n,x}(x,y)\phi_p(x,y)T_n(t)$$
$$+\left(12-13x+6x^2-x^3\right)\left(\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t)+\phi_{n,yy}(x,y)\phi_p(x,y)T_n(t)\right)\Big]$$
$$+\sum_{i=1}^{N}\Big[\frac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s)-\frac{M_i}{\mu}\Big(\phi_n(x,y)\phi_p(x,y)T_{n,tt}(t)+2c_i\phi_{n,x}(x,y)\phi_p(x,y)T_{n,t}(t)$$
$$+c_i^2\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t)\Big)\delta(x-c_it)\delta(y-s)\Big]\Big\}dA=\sum_{n=1}^{\infty}\int_A\phi_n(x,y)\phi_p(x,y)B_n(t)dA \tag{10}$$

Considering the orthogonality of $\phi_n(x,y)$, we have

$$B_n(t) = \frac{1}{P^*}\sum_{n=1}^{\infty}\int_A \Big\{R_0\left[\phi_{n,xx}(x,y)\phi_p(x,y)T_{n,tt}(t) + \phi_{n,yy}(x,y)\phi_p(x,y)T_{n,tt}(t)\right]$$
$$-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_n(x,y)\phi_p(x,y)T_n(t) + \frac{G_0}{\mu}\Big[\left(-13+12x-3x^2\right)\phi_{n,x}(x,y)\phi_p(x,y)T_n(t)$$
$$+\left(12-13x+6x^2-x^3\right)\left(\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t) + \phi_{n,yy}(x,y)\phi_p(x,y)T_n(t)\right)\Big]$$
$$+\sum_{i=1}^{N}\Big[\frac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s) - \frac{M_i}{\mu}\Big(\phi_n(x,y)\phi_p(x,y)T_{n,tt}(t)$$
$$+2c_i\phi_{n,x}(x,y)\phi_p(x,y)T_{n,t}(t) + c_i^2\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t)\Big)\delta(x-c_it)\delta(y-s)\Big]\Big\}dA \qquad (11)$$

where $P^* = \int_A \phi_p^2 dA$

Using (11), equation (6), taking into account (3) and (4), can be written as

$$\phi_n(x,y)\left[\frac{D\omega_n^4}{\mu}T_n(t) + T_{n,tt}(t)\right] = \frac{\phi_n(x,y)}{P^*}\sum_{q=1}^{\infty}\int_A \Big\{R_0\left[\phi_{q,xx}(x,y)\phi_p(x,y)T_{q,tt}(t) + \phi_{q,yy}(x,y)\phi_p(x,y)T_{q,tt}(t)\right]$$
$$-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_q(x,y)\phi_p(x,y)T_q(t) + \frac{G_0}{\mu}\Big[\left(-13+12x-3x^2\right)\phi_{q,x}(x,y)\phi_p(x,y)T_q(t)$$
$$+\left(12-13x+6x^2-x^3\right)\left(\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t) + \phi_{q,yy}(x,y)\phi_p(x,y)T_q(t)\right)\Big]$$
$$+\sum_{i=1}^{N}\Big[\frac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s) - \frac{M_i}{\mu}\Big(\phi_q(x,y)\phi_p(x,y)T_{q,tt}(t)$$
$$+2c_i\phi_{q,x}(x,y)\phi_p(x,y)T_{q,t}(t) + c_i^2\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t)\Big)\delta(x-c_it)\delta(y-s)\Big]\Big\}dA \qquad (12)$$

Equation (12) must be satisfied for arbitrary x, y and this is possible only when

$$T_{n,tt}(t) + \frac{D\omega_n^4}{\mu}T_n(t) = \frac{1}{P^*}\sum_{q=1}^{\infty}\int_A \Big\{R_0\left[\phi_{q,xx}(x,y)\phi_p(x,y)T_{q,tt}(t) + \phi_{q,yy}(x,y)\phi_p(x,y)T_{q,tt}(t)\right]$$
$$-\frac{F_0}{\mu}\left[4x-3x^2+x^3\right]\phi_q(x,y)\phi_p(x,y)T_q(t) + \frac{G_0}{\mu}\Big[\left(-13+12x-3x^2\right)\phi_{q,x}(x,y)\phi_p(x,y)T_q(t)$$
$$+\left(12-13x+6x^2-x^3\right)\left(\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t) + \phi_{q,yy}(x,y)\phi_p(x,y)T_q(t)\right)\Big]$$
$$+\sum_{i=1}^{N}\Big[\frac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s) - \frac{M_i}{\mu}\Big(\phi_q(x,y)\phi_p(x,y)T_{q,tt}(t)$$
$$+2c_i\phi_{q,x}(x,y)\phi_p(x,y)T_{q,t}(t) + c_i^2\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t)\Big)\delta(x-c_it)\delta(y-s)\Big]\Big\}dA \qquad (13)$$

The system in equation (13) is a set of coupled ordinary differential equations.
Considering the property of the Dirac-Delta function and expressing it in the Fourier cosine series as

$$\delta(x-c_it) = \frac{1}{L_X}\left[1+2\sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{j\pi x}{L_X}\right] \qquad (14)$$

and

$$\delta(y-s) = \frac{1}{L_Y}\left[1+2\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\cos\frac{k\pi y}{L_Y}\right] \qquad (15)$$

equation (13) becomes

$$\frac{d^2T_n(t)}{dt^2}+\alpha_n^2T_n(t)-\frac{1}{P^*}\sum_{q=1}^{\infty}\left\{R_0P_1^*\frac{d^2T_q(t)}{dt^2}-\left[\frac{F_0}{\mu}P_{2A}^*-\frac{G_0}{\mu}P_{2B}^*\right]T_q(t)\right.$$
$$-\sum_{i=1}^{N}\frac{M_i}{L_XL_Y\mu}\left[2\left(\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi c_it}{L_X}P_3^{***}(j)\right.\right.$$
$$\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_it}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2T_q(t)}{dt^2}+4c_i\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)\right.$$
$$\left.+\sum_{j=1}^{\infty}\cos\frac{j\pi c_it}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_it}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_q(t)}{dt} \tag{16}$$
$$+2c_i^2\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi c_it}{L_X}P_5^{***}(j)\right.$$
$$\left.\left.\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_it}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)T_q(t)\right]\right\}=\sum_{i=1}^{N}\frac{M_ig}{P^*\mu}\phi_p(c_it,s)$$

where $\alpha_n^2=\dfrac{D\omega_n^4}{\mu}$,

$P_1^*=\int_0^{L_X}\int_0^{L_Y}\left[\phi_{n,xx}(x,y)+\phi_{n,yy}(x,y)\right]\phi_p(x,y)\,dy\,dx,\ P_2^*=\int_0^{L_X}\int_0^{L_Y}\left[4x-3x^2+x^3\right]\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$

$P_3^*=\int_0^{L_X}\int_0^{L_Y}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,\quad P_3^{**}(k)=\int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$

$P_3^{***}(j)=\int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,\quad P_3^{****}(j,k)=\int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_n(x,y)\phi_p(x,y)\,dy\,dx,$

$P_4^*=\int_0^{L_X}\int_0^{L_Y}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,\quad P_4^{**}(k)=\int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,$

$P_4^{***}(j)=\int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,\quad P_4^{****}(j,k)=\int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_{n,x}(x,y)\phi_p(x,y)\,dy\,dx,$

$P_5^*=\int_0^{L_X}\int_0^{L_Y}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,\quad P_5^{**}(k)=\int_0^{L_X}\int_0^{L_Y}\cos\frac{k\pi y}{L_Y}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,$

$P_5^{***}(j)=\int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,\quad P_5^{****}(j,k)=\int_0^{L_X}\int_0^{L_Y}\cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\phi_{n,xx}(x,y)\phi_p(x,y)\,dy\,dx,$

$P_{2A}^*=4h_1-3h_2+h_3,\ P_{2B}^*=-13h_4+12h_5-3h_6+12(h_7+h_8)-13(h_9+h_{10})+6(h_{11}+h_{12})-(h_{13}+h_{14})$

$h_1=\int_0^{L_Y}\int_0^{L_X}x\phi_n(x,y)\phi_p(x,y)dxdy,\ h_2=\int_0^{L_Y}\int_0^{L_X}x^2\phi_n(x,y)\phi_p(x,y)dxdy,\ h_3=\int_0^{L_Y}\int_0^{L_X}x^3\phi_n(x,y)\phi_p(x,y)dxdy$

$h_4=\int_0^{L_Y}\int_0^{L_X}\phi_{n,x}(x,y)\phi_p(x,y)dxdy,\ h_5=\int_0^{L_Y}\int_0^{L_X}x\phi_{n,x}(x,y)\phi_p(x,y)dxdy,\ h_6=\int_0^{L_Y}\int_0^{L_X}x^2\phi_{n,x}(x,y)\phi_p(x,y)dxdy$

$h_7=\int_0^{L_Y}\int_0^{L_X}\phi_{n,xx}(x,y)\phi_p(x,y)dxdy,\ h_8=\int_0^{L_Y}\int_0^{L_X}\phi_{n,yy}(x,y)\phi_p(x,y)dxdy,\ h_9=\int_0^{L_Y}\int_0^{L_X}x\phi_{n,xx}(x,y)\phi_p(x,y)dxdy$

$h_{10}=\int_0^{L_Y}\int_0^{L_X}x\phi_{n,yy}(x,y)\phi_p(x,y)dxdy,\ h_{11}=\int_0^{L_Y}\int_0^{L_X}x^2\phi_{n,xx}(x,y)\phi_p(x,y)dxdy,\ h_{12}=\int_0^{L_Y}\int_0^{L_X}x^2\phi_{n,yy}(x,y)\phi_p(x,y)dxdy$

$h_{13}=\int_0^{L_Y}\int_0^{L_X}x^3\phi_{n,xx}(x,y)\phi_p(x,y)dxdy$ *and* $h_{14}=\int_0^{L_Y}\int_0^{L_X}x^3\phi_{n,yy}(x,y)\phi_p(x,y)dxdy$

The second order coupled differential equation (16) is the transformed equation governing the problem of a rectangular plate with general boundary condition and resting on a Pasternak elastic foundation with stiffness variation.

$\phi_n(x,y)$ are assumed to be the products of the functions $\psi_{ni}(x)$ and $\psi_{nj}(y)$ which are the beam functions in the directions of x and y axes respectively [24]. That is

$$\phi_n(x,y) = \psi_{ni}(x)\psi_{nj}(y) \tag{17}$$

these beam functions can be defined respectively, as

$$\psi_{ni}(x) = \sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X} \tag{18}$$

and $\psi_{nj}(x) = \sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}$ (19)

where A_{ni}, A_{nj}, B_{ni}, B_{nj}, C_{ni} and C_{nj} are constants determined by the boundary conditions. Ω_{ni} and Ω_{nj} are called the mode frequencies.

In order to solve equation (16) we shall consider only one mass M traveling with uniform velocity c along the line y = s. Thus for the single mass M equation (16) reduces to

$$\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\left\{R_0 P_1^*\frac{d^2T_q(t)}{dt^2} - \frac{G_0}{\mu}\left[\frac{F_0}{G_0}P_{2A}^* - P_{2B}^*\right]T_q(t)\right.$$
$$-\Gamma^0\left[2\left(\frac{P_3^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)\right.\right.$$
$$\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2T_q(t)}{dt^2} + 4c\left(\frac{P_4^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)\right.$$
$$\left.+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_q(t)}{dt}$$
$$+2c^2\left(\frac{P_5^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)\right.$$
$$\left.\left.\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)T_q(t)\right]\right\} = \frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \tag{20}$$

where $\Gamma^0 = \frac{M}{L_X L_Y \mu}$ (21)

Equation (20) is the fundamental equation of our problem when the rectangular plate has arbitrary end support conditions. We shall discuss two cases of the equation (20) namely; the **moving force** and the **moving mass** problems.

Case I: Rectangular plate traversed by a moving force

In this section, an approximate model of the differential equation describing the response of a rectangular plate resting on a variable non-Winkler (Pasternak) elastic foundation and traversed by a moving force would be obtained from equation (20) by setting $\Gamma^0 = 0$.

Thus, setting $\Gamma^0 = 0$, equation (20) reduces to

$$\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - R_0\sum_{q=1}^{\infty}\frac{P_1^*}{P^*}\frac{d^2T_q(t)}{dt^2} - \frac{G_0}{\mu}\sum_{q=1}^{\infty}\frac{1}{P^*}\left[P_{2B}^* - \frac{F_0}{G_0}P_{2A}^*\right]T_q(t) = \frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \tag{22}$$

This represents the classical case of a moving force problem associated with our system. It is an approximate model which assumes the inertia effect of the moving mass as negligible. An exact analytical solution to equation (22) is evidently not possible. Consequently, the approximate analytical solution technique, which is a modification of the asymptotic method of Struble [24] shall be used.

First, we neglect the rotatory inertial term and rearrange the equation (22) to take the form

$$\frac{d^2T_n(t)}{dt^2}+\left[\alpha_n^2-\Gamma^*\left(P_{2B}^*-\frac{F_0}{G_0}P_{2A}^*\right)\right]T_n(t)-\Gamma^*\sum_{\substack{q=1\\q\neq n}}^{\infty}\left[P_{2B}^*-\frac{F_0}{G_0}P_{2A}^*\right]T_q(t)=\frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \quad (23)$$

where

$$\Gamma^*=\frac{G_0}{\mu P^*} \quad (24)$$

By means of this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the shear modulus G_0. An equivalent free system operator defined by the modified frequency then replaces equation (23). Thus, we set the right hand side of (23) to zero and consider a parameter $\lambda^* < 1$ for any arbitrary ratio Γ^* defined as

$$\lambda^*=\frac{\Gamma^*}{1+\Gamma^*} \quad (25)$$

so that $\Gamma^*=\lambda^*+o(\lambda^{*2})$ (26)

Thus, the homogeneous part of equation (23) can be replaced with

$$\frac{d^2T_n(t)}{dt^2}+\gamma_s^2T_n(t)=0 \quad (27)$$

where

$$\gamma_s=\alpha_n-\frac{\lambda^*\left(P_{2B}^*-\frac{F_0}{G_0}P_{2A}^*\right)}{2\alpha_n} \quad (28)$$

is the modified frequency due to the effect of the shear modulus of the foundation. It is observed that when $\lambda^* = 0$, we recover the frequency of the moving force problem when the shear modulus effect of the foundation is neglected

Using equation (27), equation (22) can be written as

$$\frac{d^2T_n(t)}{dt^2}+\gamma_s^2T_n(t)-\lambda_pP_1^*\frac{d^2T_n(t)}{dt^2}-\lambda_pP_1^*\sum_{\substack{q=1\\q\neq n}}^{\infty}\frac{d^2T_q(t)}{dt^2}=\frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s) \quad (29)$$

where $\lambda_p=\frac{R_0}{P^*}$

In what follows, we seek the modified frequency corresponding to the frequency of the free system due to the presence of the effect of rotatory inertia correction factor R_0. An equivalent free system operator defined by the modified frequency then replaces equation (29). To this end, the homogenous part of equation (29) is rearranged to take the form

$$\frac{d^2T_n(t)}{dt^2}+\frac{\gamma_s^2}{1-\lambda_pP_1^*}T_n(t)-\frac{\lambda_pP_1^*}{1-\lambda_pP_1^*}\sum_{\substack{q=1\\q\neq n}}^{\infty}\frac{d^2T_q(t)}{dt^2}=0 \quad (30)$$

Now consider the parameter $\varepsilon^* < 1$ for any arbitrary ratio defined as

$$\varepsilon^*=\frac{\lambda_p}{1+\lambda_p} \quad (31)$$

It can be shown that

$$\lambda_p=\varepsilon^*+o(\varepsilon^{*2}) \quad (32)$$

Following the same argument, equation (30) can be replaced with

$$\frac{d^2T_n(t)}{dt^2}+\gamma_{sf}^2T_n(t)=0 \tag{33}$$

where $$\gamma_{sf}=\gamma_s\left[1+\frac{\varepsilon^*P_1^*}{2}\right] \tag{34}$$

represents the modified frequency corresponding to the frequency of the free system due to the presence of the rotatory inertia. It is observed that when $\varepsilon^*=0$, we recover the frequency of the moving force problem when the rotatory inertia effect is neglected.

In order to solve the non-homogenous equation (29), the differential operator which acts on $T_n(t)$ is replaced by the equivalent free system operator defined by the modified frequency γ_{sf}.
Thus the moving force problem (22) is reduced to the non-homogeneous ordinary differential equation given as

$$\frac{d^2T_n(t)}{dt^2}+\gamma_{sf}^2T_n(t)=K_m\Psi_{pi}(ct)\Psi_{pj}(s) \tag{35}$$

where

$$K_m=\frac{Mg}{P^*\mu} \tag{36}$$

Using (18), equation (35) can be written as

$$\frac{d^2T_n(t)}{dt^2}+\gamma_{sf}^2T_n(t)=K_m\Psi_{pj}(s)\left[\sin\alpha_{pi}t+A_{pi}\cos\alpha_{pi}t+B_{pi}\sinh\alpha_{pi}t+C_{pi}\cosh\alpha_{pi}t\right] \tag{37}$$

where

$$\alpha_{pi}=\frac{\Omega_{pi}c}{L_X} \tag{38}$$

When equation (37) is solved in conjunction with the initial conditions (2), one obtains expression for $T_n(t)$. Thus, in view of equation (3), one obtains

$$\begin{aligned}V(x,y,t)=&\sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty}\frac{K_m\Psi_{pj}(s)}{\gamma_{sf}[\gamma_{sf}^4-\alpha_{pi}^4]}\{[\gamma_{sf}^2-\alpha_{pi}^2][C_{pi}\gamma_{sf}(\cosh\alpha_{pi}t-\cos\gamma_{sf}t)\\&+B_{pi}(\gamma_{sf}\sinh\alpha_{pi}t-\alpha_{pi}\sin\gamma_{sf}t)]+[\gamma_{sf}^2+\alpha_{pi}^2][A_{pi}\gamma_{sf}(\cos\alpha_{pi}t-\cos\gamma_{sf}t)\\&-(\alpha_{pi}\sin\gamma_{sf}t-\gamma_{sf}\sin\alpha_{pi}t)]\}[\sin\frac{\Omega_{ni}x}{L_X}+A_{ni}\cos\frac{\Omega_{ni}x}{L_X}+B_{ni}\sinh\frac{\Omega_{ni}x}{L_X}\\&+C_{ni}\cosh\frac{\Omega_{ni}x}{L_X}][\sin\frac{\Omega_{nj}y}{L_Y}+A_{nj}\cos\frac{\Omega_{nj}y}{L_Y}+B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y}+C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}]\end{aligned} \tag{39}$$

as the transverse displacement response to a moving force of a rectangular plate resting on variable non-Winkler (Pasternak) elastic foundation and having arbitrary edge supports.

Case II: Rectangular plate traversed by a moving mass

In this section, we seek the solution to the entire equation (20) when no term of the coupled differential equation is neglected.

Evidently, an exact analytical solution to equation (20) does not exist, an analytical approximate method is therefore desirable. To this end, the approximate analytical solution method of Struble that has been used to tackle this form of coupled differential equation shall be employed to treat equation (20). We take note that, neglecting the terms representing the inertia effect of the moving mass we obtain equation (29) and then equation (35). The homogeneous part of this equation can be replaced by a free system operator defined by the modified frequency γ_{sf}, due to the presence of the effects of rotatory inertia and the shear modulus of the foundation. Thus, equation (20) can be rewritten in the form

$$\left[1+\frac{2\varepsilon_s}{P^*}\left(\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\right]\frac{d^2T_n(t)}{dt^2}$$

$$+\frac{4\varepsilon_s c}{P^*}\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_n(t)}{dt}$$

$$+\left[\gamma_{sf}^2+\frac{2\varepsilon_s c^2}{P^*}\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)\right]T_n(t)$$

$$+\frac{\varepsilon_s}{P^*}\sum_{\substack{q=1\\q\neq n}}\left[2\left(\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2T_q(t)}{dt^2}\right.$$

$$4c\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_q(t)}{dt}$$

$$\left.+2c^2\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)T_q(t)\right]$$

$$=\frac{\varepsilon_s gL_X L_Y}{P^*}\Psi_{pi}(ct)\Psi_{pj}(s) \qquad (40)$$

where $$\varepsilon_s=\frac{M}{L_X L_Y \mu} \qquad (41)$$

Furthermore, for our plate model, resting on variable non-Winkler elastic foundation and traversed by a moving mass, we rearrange equation (40) to take the form

$$\frac{d^2T_n(t)}{dt^2}+\frac{\varepsilon_s R_2(t)}{1+\varepsilon_s R_1(t)}\frac{dT_n(t)}{dt}+\frac{\gamma_{sf}^2+\varepsilon_s R_3(t)}{1+\varepsilon_s R_1(t)}T_n(t)$$

$$+\frac{\varepsilon_s}{1+\varepsilon_s R_1(t)}\sum_{\substack{q=1\\q\neq n}}^{\infty}\left[R_1(t)\frac{d^2T_q(t)}{dt^2}+R_2(t)\frac{dT_q(t)}{dt}\right. \qquad (42)$$

$$\left.+R_3(t)T_q(t)\right]=\frac{\varepsilon_s gL_X L_Y}{[1+\varepsilon_s R_1(t)]P^*}\Psi_{pi}(ct)\Psi_{pj}(s)$$

where $$R_1(t)=\frac{2}{P^*}\left[\frac{P_3^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right] \qquad (43)$$

$$R_2(t)=\frac{2c}{P^*}\left[\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right] \qquad (44)$$

$$R_3(t)=\frac{2c^2}{P^*}\left[\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right] \qquad (45)$$

Going through the same arguments and analysis as in the previous case, considering the homogeneous part of equation (42), the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass M is obtained and an equivalent free system operator defined by the modified frequency then replaces equation (42).

Thus, equation (42) becomes

$$\frac{d^2T_n(t)}{dt^2}+\beta_{sf}^2T_n(t)=\frac{\mu_0 gL_X L_Y}{P^*}\Psi_{pi}(ct)\Psi_{pj}(s) \qquad (46)$$

where ε_s has been written as a function of the mass ratio μ_0 and

$$\beta_{sf} = \gamma_{sf}\left[1-\frac{\mu_0}{2}\left(R_1-\frac{R_3}{\gamma_{sf}^2}\right)\right] \tag{47}$$

is the modified frequency corresponding to the frequency of the free system due to the presence of moving mass. Here, it is remarked that this modified frequency has in it the effects of the shear modulus of the foundation and rotatory inertia. It is observed that when $\mu_0 = 0$ in equation (47), we recover the frequency of the moving force problem of the same dynamical system.

Using (18), equation (46) becomes

$$\frac{d^2T_n(t)}{dt^2}+\beta_{sf}^2T_n(t) = G_g\Psi_{pj}(s)[\sin\alpha_{pi}t + A_{pi}\cos\alpha_{pi}t + B_{pi}\sinh\alpha_{pi}t + C_{pi}\cosh\alpha_{pi}t] \tag{48}$$

where $G_g = \dfrac{\mu_0 g L_X L_Y}{P^*}$ (49)

It is noticed that equation (48) is analogous to equation (37) with β_{sf} and G_g replacing γ_{sf} and K_m respectively.

Therefore, one obtains

$$\begin{aligned} V(x,y,t) = \sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty} \frac{G_g\Psi_{pj}(s)}{\beta_{sf}[\beta_{sf}^4-\alpha_{pi}^4]} \Big\{ & [\beta_{sf}^2-\alpha_{pi}^2][C_{pi}\beta_{sf}(\cosh\alpha_{pi}t - \cos\beta_{sf}t) \\ & + B_{pi}(\beta_{sf}\sinh\alpha_{pi}t - \alpha_{pi}\sin\beta_{sf}t)] + [\beta_{sf}^2+\alpha_{pi}^2][A_{pi}\beta_{sf}(\cos\alpha_{pi}t-\cos\beta_{sf}t) \\ & - (\alpha_{pi}\sin\beta_{sf}t - \beta_{sf}\sin\alpha_{pi}t)] \Big\}[\sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} \\ & + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X}][\sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}] \end{aligned} \tag{50}$$

Equation (50) is the transverse displacement response to a moving mass of a rectangular plate resting on variable Pasternak elastic foundation and having arbitrary edge supports.

The constants A_{ni}, A_{pi}, A_{nj}, A_{pj}, B_{ni}, B_{pi}, B_{nj}, B_{pj}, C_{ni}, C_{pi}, C_{nj} and C_{pj} are to be determined from the choice of the end support condition.

4.0 Illustrative Examples

In this section, the foregoing analysis is illustrated by various practical examples. In Particular, classical boundary conditions such as Clamped-Simple end conditions and Free-Clamped end conditions are considered.

4.1 Clamped-Simple Rectangular plate

For a rectangular plate clamped at edges y = 0, y = L_Y with simple supports at edges x = 0, x = L_X, the conditions at such opposite edges are expressed as

$$V(0,y,t)=0,\ V(L_X,y,t)=0,\ V(x,0,t)=0,\qquad V(x,L_Y,t)=0 \tag{51}$$

$$\frac{\partial^2 V(0,y,t)}{\partial x^2}=0,\ \frac{\partial^2 V(L_X,y,t)}{\partial x^2}=0,\ \frac{\partial V(x,0,t)}{\partial y}=0,\ \frac{\partial V(x,L_Y,t)}{\partial y}=0 \tag{52}$$

and for normal modes

$$\Psi_{ni}(0)=0,\ \Psi_{ni}(L_X)=0,\ \Psi_{nj}(0)=0,\ \Psi_{nj}(L_Y)=0 \tag{53}$$

$$\frac{\partial^2\Psi_{ni}(0)}{\partial x^2}=0,\ \frac{\partial^2\Psi_{ni}(L_X)}{\partial x^2}=0,\ \frac{\partial\Psi_{nj}(0)}{\partial y}=0,\ \frac{\partial\Psi_{nj}(L_Y)}{\partial y}=0 \tag{54}$$

For the clamped edges, it is straight forward to show that

$$A_{nj} = \frac{\sinh\Omega_{nj}-\sin\Omega_{nj}}{\cos\Omega_{nj}-\cosh\Omega_{nj}},\quad \Rightarrow\quad A_{pj} = \frac{\sinh\Omega_{pj}-\sin\Omega_{pj}}{\cos\Omega_{pj}-\cosh\Omega_{pj}} \tag{55}$$

$$B_{nj} = -1 \Rightarrow B_{pj} = -1 \text{ and} \tag{56}$$

$$C_{nj} = -A_{nj} \Rightarrow C_{pj} = -A_{pj} \tag{57}$$

and the frequency equation of the clamped edges is given by the following determinant equation

$$\begin{vmatrix} (\sinh\Omega_{nj} - \sin\Omega_{nj}) & (\cos\Omega_{nj} - \cosh\Omega_{nj}) \\ (\cos\Omega_{nj} - \cosh\Omega_{nj}) & (\sin\Omega_{nj} + \sinh\Omega_{nj}) \end{vmatrix} = 0 \tag{58}$$

which yields $\cos\Omega_{nj}\cosh\Omega_{nj} = 1$ (59)

and for the simple edges, it is readily shown that

$A_{ni} = 0$, $B_{ni} = 0$, $C_{ni} = 0$, and $\Omega_{ni} = n_i\pi$ (60)

Similarly, $A_{pi} = 0$, $B_{pi} = 0$, $C_{pi} = 0$, and $\Omega_{pi} = p_i\pi$ (61)

Using (55), (56), (57), (59), (60) and (61) in equations (39) and (50) one obtains the displacement response respectively to a moving force and a moving mass of a simple-clamped rectangular plate resting on a variable Pasternak elastic foundation.

4.2 Clamped-Free rectangular plate

For a rectangular plate clamped at edges y = 0, y = L_Y and free at edges x = 0, x = L_X, the conditions at such opposite edges are expressed as

$$\frac{\partial^2 V(0,y,t)}{\partial x^2} = 0,\ \frac{\partial^2 V(L_X,y,t)}{\partial x^2} = 0,\ V(x,0,t) = 0,\ V(x,L_Y,t) = 0 \tag{62}$$

$$\frac{\partial^3 V(0,y,t)}{\partial x^3} = 0,\ \frac{\partial^3 V(L_X,y,t)}{\partial x^3} = 0,\ \frac{\partial V(x,0,t)}{\partial y} = 0,\ \frac{\partial V(x,L_Y,t)}{\partial y} = 0 \tag{63}$$

and for normal modes

$$\frac{\partial^2 \Psi_{ni}(0)}{\partial x^2} = 0,\ \frac{\partial^2 \Psi_{ni}(L_X)}{\partial x^2} = 0,\ \Psi_{nj}(0) = 0,\ \Psi_{nj}(L_Y) = 0 \tag{64}$$

$$\frac{\partial^3 \Psi_{ni}(0)}{\partial x^3} = 0,\frac{\partial^3 \Psi_{ni}(L_X)}{\partial x^3} = 0,\ \frac{\partial \Psi_{nj}(0)}{\partial y} = 0,\ \frac{\partial \Psi_{nj}(L_Y)}{\partial y} = 0 \tag{65}$$

Our initial conditions are of the form

$$W(x,y,0) = 0 = \frac{\partial W(x,y,0)}{\partial t} \tag{66}$$

Thus, we have

$$\begin{aligned} A_{ni} &= \frac{\sin\Omega_{ni} - \sinh\Omega_{ni}}{\cosh\Omega_{ni} - \cos\Omega_{ni}} = \frac{\cos\Omega_{ni} - \cosh\Omega_{ni}}{\sin\Omega_{ni} + \sinh\Omega_{ni}}, \quad \Rightarrow \quad A_{pi} = \frac{\sin\Omega_{pi} - \sinh\Omega_{pi}}{\cosh\Omega_{pi} - \cos\Omega_{pi}} \\ A_{nj} &= \frac{\sinh\Omega_{nj} - \sin\Omega_{nj}}{\cos\Omega_{nj} - \cosh\Omega_{nj}} = \frac{\cos\Omega_{nj} - \cosh\Omega_{nj}}{\sin\Omega_{nj} + \sinh\Omega_{nj}}, \quad \Rightarrow \quad A_{pj} = \frac{\sinh\Omega_{pj} - \sin\Omega_{pj}}{\cos\Omega_{pj} - \cosh\Omega_{pj}} \end{aligned} \tag{67}$$

$$B_{ni} = 1,\ \Rightarrow\ B_{pi} = 1,\ B_{nj} = 1 \Rightarrow B_{pj} = 1 \tag{68}$$

$$C_{ni} = A_{ni} \Rightarrow C_{pi} = A_{pi},\ C_{nj} = -A_{nj} \Rightarrow C_{pj} = -A_{pj} \tag{69}$$

and the frequency equation for the dynamical problem is

$$\cos\Omega_{ni}\cosh\Omega_{ni} = 1 \text{ and } \cos\Omega_{nj}\cosh\Omega_{nj} = 1 \tag{70}$$

such that $\Omega_{1i} = 4.73004,\ \Omega_{2i} = 7.85320,\ \Omega_{3i} = 10.99561......$ and (71a)

$$\Omega_{1j} = 4.73004,\ \Omega_{2j} = 7.85320,\ \Omega_{3j} = 10.99561...... \tag{71b}$$

Using (67), (68), (69), (71a) and (71b) in equations (39) and (50) one obtains the displacement response respectively to a moving force and a moving mass of a free-clamped rectangular plate resting on a variable Pasternak elastic foundation.

5.0 Discussion of the Analytical Solutions

Here, we shall examine the phenomenon of resonance. From equation (39), it is evident that the rectangular plate on a variable Pasternak elastic foundation and traversed by a moving force encounters a resonance effect when

$$\gamma_{sf} = \frac{\Omega_{pi} c}{L_X} \tag{72}$$

while equation (50) reveals that the same plate under the action of a moving mass reaches the state of resonance whenever

$$\beta_{sf} = \frac{\Omega_{pi} c}{L_X} \tag{73}$$

where
$$\beta_{sf} = \gamma_{sf}\left[1 - \frac{\mu_0}{2}\left(R_1 - \frac{R_3}{\gamma_{sf}^2}\right)\right] \tag{74}$$

Equations (73) and (74) imply

$$\gamma_{sf}\left[1 - \frac{\mu_0}{2}\left(R_1 - \frac{R_3}{\gamma_{sf}^2}\right)\right] = \frac{\Omega_{pi} c}{L_X} \tag{75}$$

Consequently, for the same natural frequency, the critical speed (and the natural frequency) for the moving mass problem is smaller than that of the moving force problem. Thus, resonance is reached earlier in the moving mass system than in the moving force system.

6.0 Numerical Calculations and Discussion of Results

Calculations of practical interests in dynamics of structures are presented in this section for all the illustrative examples. A rectangular plate of length L_Y = 0.914m and breadth L_X = 0.457m has been considered. The mass is assumed to travel at the constant velocity 0.8123m/s. Also, E, S and Γ are chosen to be 2.109×10^9 kg/m^2, 0.4m and 0.2 respectively. The results are as presented on the various graphs below for the various classes of boundary conditions.

6.1 Clamped-Simple rectangular plate

The dynamic responses of the rectangular plate clamped at the edges y = 0 and y = L_Y and simply supported at the edges x = 0 and x = L_X are presented in Figures 1and 2.

Figure 1 displays the effect of Rotatory inertia (R_0) on the transverse deflection of the clamped-simple rectangular plate for the case of moving force. The graph show that the response amplitudes decrease as the value of the Rotatory inertia correction factor increases. Figure 2 displays the effect of Shear modulus (G_0) on the transverse deflection of the clamped-simple rectangular plate for the case of moving mass. It is shown that the response amplitudes decrease as the value of G_0 increases.

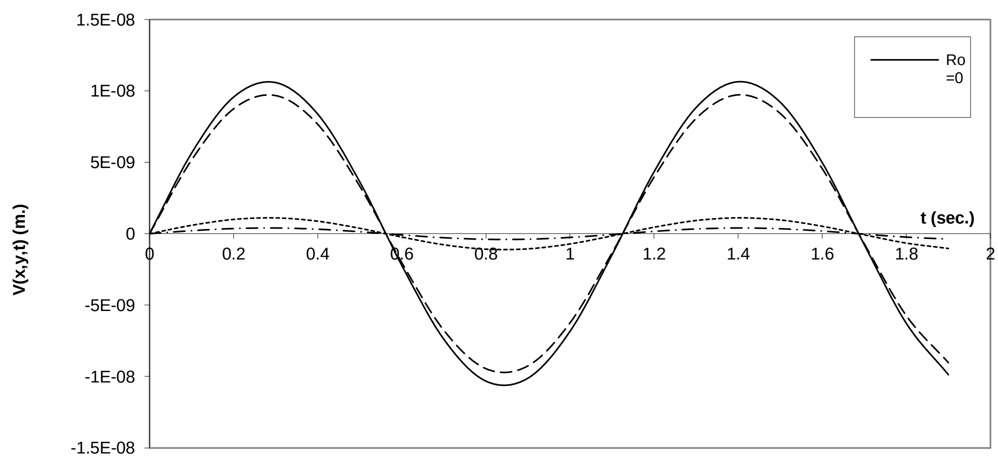

Figure 1: Deflection profile of clamped-simple plate resting on variable Pasternak foundation and traversed by moving force for Fo=1000000, Go=900000 and various values of Ro.

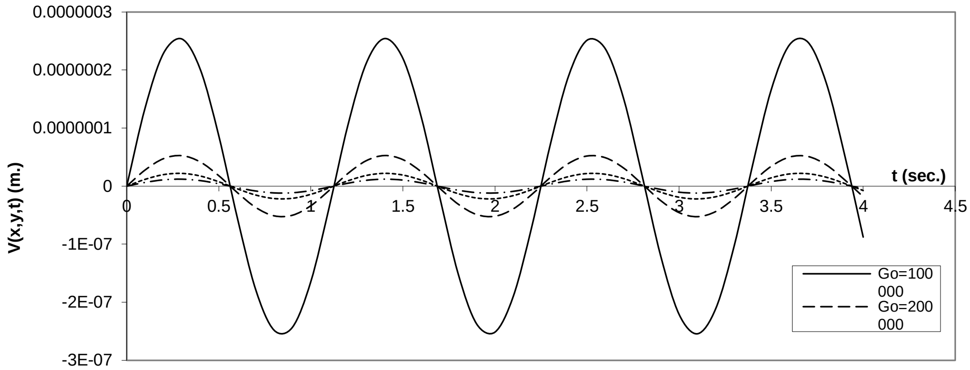

Figure 2: Displacement profile of clamped-simple rectangular plate resting on variable Pasternak foundation and traversed by moving mass for Fo=1000000, Ro=0.4 and various values of Go.

6.2 Clamped-Free rectangular plate

The dynamic responses of the rectangular plate clamped at the edges y = 0 and y = L_Y and free at the edges x = 0 and x = L_X are presented in Figures 3-5.

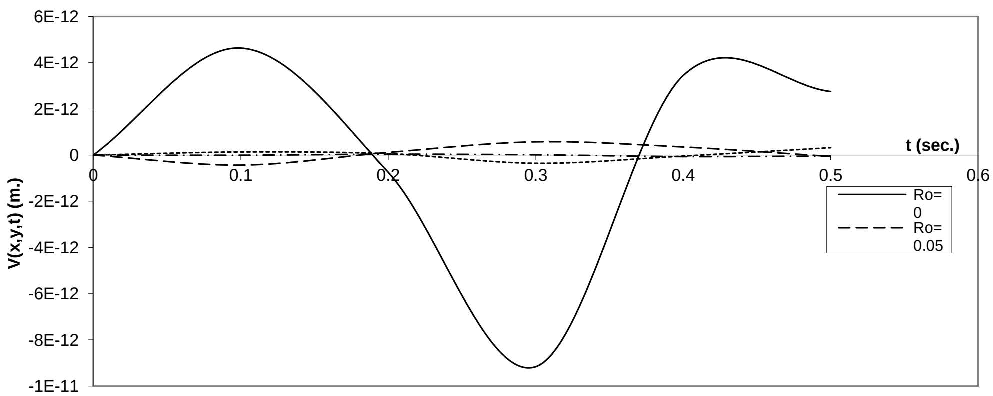

Figure 3: Displacement profile of clamped-free plate on variable Pasternak foundation and traversed by moving mass for Fo=2000000, Go=1000 and various values of Ro.

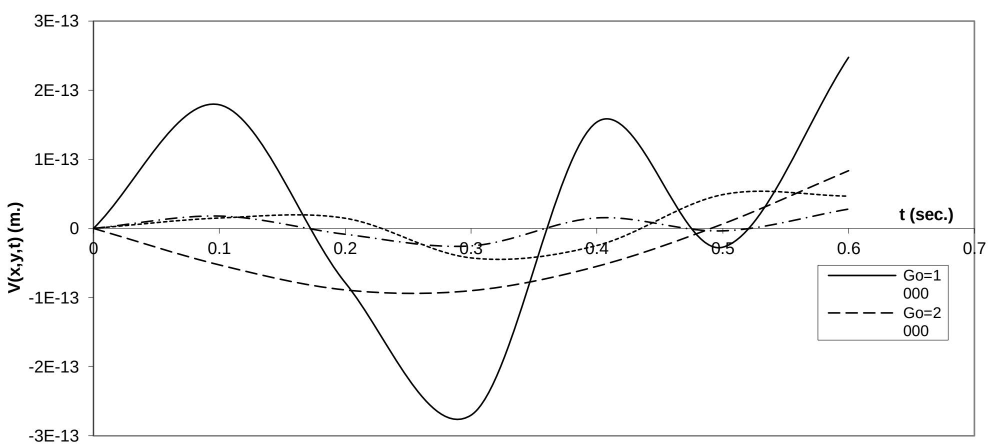

Figure 4: Deflection profile of clamped-free rectangular plate on variable Pasternak foundation and traversed by moving force for Fo=100000, Ro=0.4 and various values of Go.

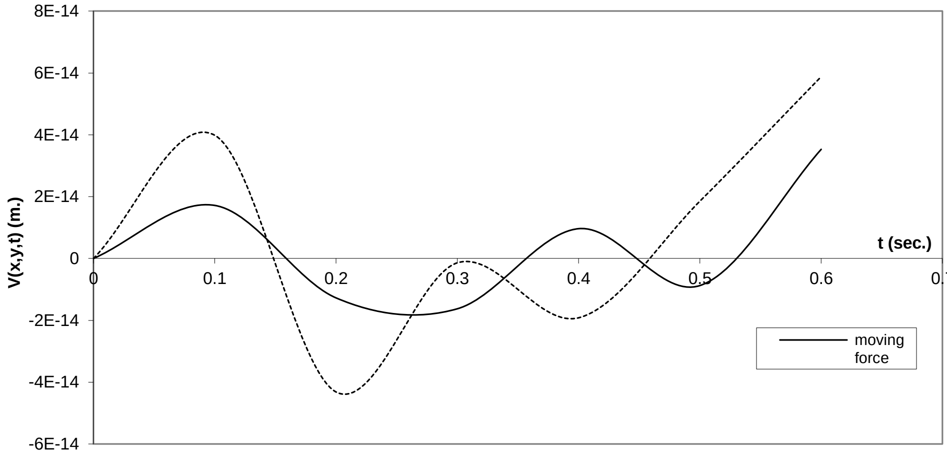

Figure 5: Comparison of the deflections of moving force and moving mass cases for clamped-free rectangular plate resting on variable Pasternak foundation with Fo=1000000, Ro=0.4 and Go=1000.

Figures 3 and 4 display the effects of Rotatory inertia (R_0) and Shear modulus (G_0) respectively on the transverse deflection of the clamped-free rectangular plate for the cases of moving mass and moving force respectively. The graphs show that the response amplitudes decrease as the values of R_0 and G_0 increase.

Figure 5 compares the displacement curves of the moving force and moving mass for a clamped-free rectangular plate with F_0 = 1000000 N/m^3, R_0 = 0.4 and G_0 = 100000 N/m. Evidently, the response amplitude of the moving mass is greater than that of the moving force problem.

7.0 Conclusion

The problem of the dynamic analysis of rectangular plates with general classical boundary conditions and resting on variable Pasternak elastic foundations has been studied in this work. The closed form solutions of the fourth order partial differential equations with variable and singular coefficients of the rectangular plate is obtained for both cases of moving force and moving mass. The solution technique is based on the technique of Shadnam et al [13] which was used to remove the singularity in the governing fourth order partial differential equation and to reduce it to a sequence of coupled second order differential equations. These coupled second order differential equations were then simplified using the modified Struble's asymptotic technique. The methods of integral transformation and the convolution theory are then employed to obtain the analytical solution of the two-dimensional dynamical problem.

These solutions are analyzed and resonance conditions are obtained for the problem. The analyses carried out show that the moving force solution is not an upper bound for the accurate solution of the moving mass problem and that as the rotatory inertia correction factor increases, the response amplitudes of the plates decrease for both cases of moving force and moving mass problem. When the rotatory inertia correction factor is fixed, the displacements of the rectangular plates resting on variable Pasternak elastic foundations decrease as the shear modulus increases for the variants of classical boundary conditions considered. Also, as the foundation modulus increases, the response amplitudes of the plates decrease, the effect of shear modulus is observed to be more noticeable than that of the foundation modulus.

It is shown further from the results that, for fixed values of rotatory inertia correction factor, foundation modulus and shear modulus, the response amplitude for the moving mass problem is greater than that of the moving force problem implying that resonance is reached earlier in moving mass problem than in moving force problem of the rectangular plate resting on variable Pasternak elastic foundation for the variants of boundary condition considered. Also, an increase in the shear modulus results in an increase in the critical speed of the moving load; this shows that risk is reduced when the shear modulus increases. The same result obtains for an increase in both foundation modulus and rotatory inertial correction factor for all the classical boundary conditions that are considered.

Finally, the results in this work agree with what obtains in literature [6,9 and 10]. Hence the method employed in this work is accurate and the solutions are convergent.

References.

[1] **Fryba, L. (1972):** Vibration of solids and structures under moving loads. Groningen: Noordhoff.

[2] **Oni, S. T. (1991):** On the dynamic response of elastic structures to moving multi-mass system. Ph.D. thesis. University of Ilorin, Nigeria.

[3] **Inglis, C. E. (1934):** A mathematical treatise on vibration in railway bridges. The University press, Cambridge.

[4] **Gbadeyan, J. A. and Aiyesimi, Y. M. (1990):** Response of an elastic beam resting on viscoelastic foundation to a load moving at non-uniform speed. Nigerian Journal of Mathematics and Applications, Vol.3, pp 73-90.

[5] **Sadiku, S. and Leipholz, H. H. E. (1981):** On the dynamics of elastic systems with moving concentrated masses. Ing. Archiv. 57, 223-242.

[6] **Gbadeyan, J. A. and Oni, S. T. (1995):** Dynamic behaviour of beams and rectangular plates under moving loads. Journal of Sound and Vibration. 182(5), pp 677-695.

[7] **Franklin, J. N. and Scott, R. F. (1979):** Beam equation with variable foundation coefficient. ASCE. Eng. Mech. Div. Vol. 105, EMS, pp 811-827.

[8] **Lentini, M. (1979):** Numerical solution of the beam equation with non-uniform foundation coefficient. ASME. Journal of Applied Mechanics.Vol.46.pp 901-902.

[9] **S. T. Oni and T. O. Awodola. (2005):** Dynamic response to moving concentrated masses of uniform Rayleigh beams resting on variable Winkler elastic foundation. Journal of the Nigerian Association of Mathematical Physics. Vol. 9. pp 151-162.

[10] **S. T. Oni, and T. O. Awodola (2010):** Dynamic response under a moving load of an elastically supported non-prismatic Bernoulli-Euler beam on variable elastic foundation. Latin American Journal of Solids and Structures. 7, pp 3 - 20.

[11] **Moshe Eisenberger and Jose Clastornik. (1987):** Beams on variable Two-Parameter elastic foundation. Journal of Engineering Mechanics, Vol. 113, No. 10, pp 1454-1466.

[12] **Gbadeyan, J. A. and Oni, S. T. (1992):** Dynamic response to a moving concentrated masses of elastic plates on a non-Winkler elastic foundation. Journal of Sound and Vibration, 154, pp 343-358.

[13] **M. R. Shadnam, M. Mofid and J. E. Akin (2001):** On the dynamic response of rectangular plate, with moving mass. Thin-Walled Structures, 39(2001), pp 797 – 806.

[14] **S. T. Oni (2004):** Flexural motions of a uniform beam under the actions of a concentrated mass traveling with variable velocity. Abacus, Journal of Mathematical Association of Nigeria. Vol. 31, No 2a, pp 79 – 93.

[15] **Oni S. T. and Omolofe B. (2005):** Dynamic analysis of a prestressed elastic beam with general boundary conditions under moving loads at varying velocities. Journal of Engineering and Engineering Technology,FUTA,Vol. 4,No1, pp55–72.

[16] **Oni S. T. and Awodola T. O. (2003):** Vibrations under a moving load of a non-uniform Rayleigh beam on variable elastic foundation. Journal of Nigerian Association of Mathematical Physics, Vol. 7, pp 191 – 206.

[17] **Omer Civalek and Aitung Yauas (2006):** Large deflection static analysis of rectangular plates on two parameter elastic foundations. International Journal of Science and Technology. Vol. 1, No. 1, pp 43 – 50.

[18] **G. G. Adams (1995):** Critical speeds and the response of a tensioned beam on an elastic foundation to repetitive moving loads. Int. Jour. Mech. Sci., Vol. 7, pp 773 – 781.

[19] **E. Savin (2001):** Dynamics amplification factor and response spectrum for the evaluation of vibrations of beams under successive moving loads. Journal of Sound and Vibrations, 248(2), pp 267 – 288.

[20] **S. T. Oni and T. O. Awodola (2011):** Dynamic behaviour under moving concentrated masses of simply supported rectangular plates resting on variable Winkler elastic foundation. Latin American Journal of Solids and Structures (LAJSS), vol. 8, pp 373-392.

[21] **Jia-Jang Wu (2006):** Vibration analysis of a portal frame under the action of a moving distributed mass using moving mass element. Int. Jour. for Numerical Methods in Engineering, 62, pp 2028 – 2052.

[22] **Jaeger, J. C. and Strarfield, A. M. (1979):** An introduction to applied Mathematics. Oxford University Press, pp 295-296.

[23] **S. T. Oni and T. O. Awodola (2012):** Vibrations of a simply supported plate under moving masses and resting on Pasternak elastic foundation with stiffness variation. Journal of the Nigerian Mathematical society (Accepted in May, 2012).

[24] **S. T. Oni and T. O. Awodola (2012):** Dynamic response to moving masses of rectangular plates with general boundary conditions and resting on variable Winkler foundation. Latin American Journal of Solids and Structures (Accepted in September, 2012).