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On the Degree of Accuracy Between Two Characteristics in Laminar and Turbulent Boundary Layer Flows.

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Abstract

In this work we consider two cases. In case one, we apply a linear velocity profile in the energy thickness of Wieghardt [1] and also in Karman-Pohlhausen [2] integral momentum equation to enable us determine two approximate characteristics of interest in the laminar boundary layer flow, namely: boundary layer thickness and skin friction coefficient and their values are compared with the exact Blasius [3] values. In case two, a one-seventh power law velocity profile is incorporated into the Pohlhausen model for the determination of the same characteristics in the turbulent boundary layer and their values are also compared with exact values. In each case the error of each characteristics is determined. By comparing the two characteristics in the two cases for their errors the accuracy (or otherwise) of these characteristics is determined.

Keywords: Degree of accuracy, approximate characteristics, laminar, turbulent, boundary layer flows.

1.0 Introduction

The subject, boundary layer theory, has been discussed extensively by many authors since the development of the concept by Prandtl [4]. For instance, Habib et al [5] carried out transient calculation of the boundary layer flow over spills using simulation and experimental approaches. They validated their results against experimental data and also made comparison of the simulated results with empirical prediction model.

Craft and Lowell [6] applied steady state boundary layer theory to two aspects of oceanic hydrothermal heat flux. In their analysis they showed that, for near-axis model, heat transfer in the hydrothermal boundary layer is greater then the input from steady state generation of the oceanic crust by sea flow spreading.

Dorfman [7] presented a review of universal functions widely used in different areas of boundary layer theory for many years up to the present. In his work he adopted various solutions from many published articles to show the breadth of universal approaches with application in laminar, turbulent and transition boundary layers in solving non-isothermal and conjugate heat transfer problems as well as in planetary boundary layer problems in meteorology.

There are several other contributors to the concept of laminar and turbulent boundary layer flows. Notable among them include: Batchelor [8], Schlichting [9], Afzal [10], Vyas and Ranjan [11], Mohmoudian and Scales [12], Eyo [13], etc.

In this work two velocity profiles are employed, one for laminar flow used in energy thickness of Wieghardt [1] and in Pohlhausen [2] momentum integral equation, and the other for turbulent flow used also in Pohlhausen [2] model to be able to determine the approximate values of two characteristics in the laminar and turbulent zones. Their values are compared with the exact values and the degree of accuracy determined.

2.0 Boundary Layer Equations for Laminar Flow

For steady, two- dimensional laminar flow, the boundary layer equations together with boundary conditions are as follows:

Continuity equation:
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2.1)$$

Bernoulli equation:
$$\frac{dp}{dx}=-\rho U\frac{dU}{dx} \qquad (2.2)$$

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Navies- States equation: $$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\upsilon\frac{\partial^2 u}{\partial y^2} \qquad (2.3)$$

B.Cs: $$\left.\begin{aligned} &u=v=0 \ \ at \ \ y=0 \\ &\frac{\partial u}{\partial y}=0 \ \ at \ \ y=\delta \end{aligned}\right\} \qquad (2.4)$$

where x, y are space coordinates, u, v are the corresponding velocity components, U the free stream velocity, ρ the fluid density, p the fluid pressure, $\upsilon = \mu/\rho$ is the kinematic viscosity while μ is the dynamic viscosity.
With the above equations, the energy thickness of Wieghardt [1] can be derived in the form

$$\frac{\rho}{2}\ \frac{d}{dx}\ \left(U^3\delta_3(x)\right)=\mu\int_0^{\delta}\left(\frac{\partial\ u}{\partial y}\right)^2 dy \qquad (2.5)$$

(see Eyo [13] and Schlichting [9]).

2.1 Exact Blasius Values for Boundary Layer Thickness and Skin Friction Coefficient for Laminar Boundary Layer Flow

Following Blasius [3], the following results were obtained:

(i) Boundary layer thickness δ

$$\frac{\delta}{x}=\frac{5}{(\mathrm{Re}_x)^{1/2}} \qquad (2.6)$$

(ii) Skin Friction Coefficient C_f

$$C_f=\frac{0.664}{(\mathrm{Re}_x)^{1/2}} \qquad (2.7)$$

2.2 Exact Blasius values for Boundary Layer Thickness, Viscous Shear Stress and Skin Friction Coefficient for Turbulent Boundary Layer Flow.

Similarly, the following results were derived by Blasius [3]:

(iii) Boundary layer thickness δ

$$\frac{\delta}{x}=\frac{0.382}{(\mathrm{Re}_x)^{1/5}} \qquad (2.8)$$

(iv) Viscous shear stress near the plate τ_0

$$\tau=0.0226\ \rho\, U^2\left(\frac{\mu}{\rho\, U\delta}\right)^{1/4} \qquad (2.9)$$

(v) Skin friction coefficient C_f

$$C_f=\frac{0.0574}{(\mathrm{Re}_x)^{1/5}} \qquad (2.10)$$

3.0 Determination of Approximate Characteristics in Laminar Boundary Layer Flow

(i) Boundary layer thickness δ

We begin by first applying a linear velocity profile of the form

$$\frac{u}{U}=\frac{y}{\delta} \qquad (3.1)$$

in the energy thickness equation (2.5). Now, from (3.1) we find

$$\frac{\partial u}{\partial y}=\frac{U}{\delta}$$

and

$$\left(\frac{\partial u}{\partial y}\right)^2=\frac{U^2}{\delta^2} \qquad (3.2)$$

Substituting (3.2) in the rhs of (2.5) we have

$$\frac{\rho}{2}\frac{d}{dx}\left(U^3\delta_3(x)\right)=\mu\int_0^\delta\frac{U^2}{\delta^2}dy \qquad (3.3)$$

Integrating the rhs of (3.3), we find after simplification

$$\frac{\rho}{2}\frac{d}{dx}\left(U^3\delta_3(x)\right)=\frac{\mu U^2}{\delta} \qquad (3.4)$$

ie

$$\frac{d}{dx}\delta_3=\frac{2}{\rho\, U^3}\cdot\frac{\mu U^2}{\delta} \qquad (3.5)$$

or

$$\frac{d}{dx}\delta_3=\frac{2\mu}{\rho\, U\delta} \qquad (3.6)$$

Next, we apply (3.1) in Pohlhausen [2] integral equation for energy thickness given in the form

$$\delta_3(x)=\int_0^\delta\frac{u}{U}\left[1-\left(\frac{u}{U}\right)^2\right]dy \qquad (3.7)$$

Substituting (3.1) in (3.7) we have

$$\delta_3(x)=\int_0^\delta\left[\frac{y}{\delta}-\left(\frac{y}{\delta}\right)^3\right]dy \qquad (3.8)$$

Integrating (3.8) we find

$$\delta_3(x)=\left[\frac{y^2}{2\delta}-\frac{y^4}{4\delta^3}\right]_{y=0}^{y=\delta} \qquad (3.9)$$

which on simplification gives

$$\delta_3(x)=\frac{\delta}{4} \qquad (3.10)$$

or

$$\frac{d}{dx}\delta_3=\frac{1}{4}\frac{d}{dx}\delta \qquad (3.11)$$

Comparing (3.6) and (3.11), we find

$$\frac{1}{4}\frac{d}{dx}\delta=\frac{2\mu}{\rho U\delta} \qquad (3.12)$$

i.e

$$\frac{1}{8}\frac{d}{dx}\left(\delta^2\right)=\frac{2\mu}{\rho U} \qquad (3.13)$$

or

$$\frac{d}{dx}\left(\delta^2\right)=\frac{16\mu}{\rho U} \qquad (3.14)$$

Integration of (3.14) results in

$$\delta^2=16\frac{\mu x}{\rho U}+C \qquad (3.15)$$

where C is the constant of integration. We note that when $x=0,\delta=0$ and C = 0. Hence equation (3.15) becomes

$$\delta=4x\sqrt{\frac{\mu}{\rho Ux}} \qquad (3.16)$$

or

$$\frac{\delta}{x}=\frac{4}{\left(\mathrm{Re}_x\right)^{1/2}} \qquad (3.17)$$

(ii) Skin friction coefficient C_f

The wall shear stress τ_0 is given by

$$\tau_0=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (3.18)$$

Applying (3.1) in (3.18) we find

$$\tau_0=\frac{\mu U}{\delta} \qquad (3.19)$$

Thus, the skin friction coefficient C_f is given by Schlichting [9]

$$C_f=\frac{\tau_0}{\frac{1}{2}\rho U^2} \qquad (3.20)$$

Substituting (3.19) in (3.20) and simplifying we have

$$C_f=\frac{2\mu}{\rho U\delta} \qquad (3.21)$$

i.e

$$C_f=\frac{2\mu x}{\rho Ux}\delta^{-1} \qquad (3.22)$$

Using (3.17) in (3.22) we obtain after simplification

$$C_f=\frac{0.5}{\left(\mathrm{Re}_x\right)^{1/2}} \qquad (3.23)$$

4.0 Determination of Approximate Characteristics in Turbulent Boundary Layer Flow

(i) Boundary layer thickness δ

The integral momentum equation is given by Schlichting [9]

$$\frac{d}{dx}\int_0^{\delta} u(U-u)dy = \frac{\tau_o}{\rho} \qquad (4.1)$$

while the velocity profile typical of turbulent boundary layer is of the form

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \qquad (4.2)$$

for Reynolds number satisfying $5\times10^5 < \mathrm{Re} < 10^7$

Substituting (4.2) and (2.9) into (4.1) we find

$$\frac{d}{dx}\int_o^{\delta} U\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\left[u\left(\frac{y}{\delta}\right)^{-\frac{1}{7}} - U\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right]dy = \frac{1}{\rho}\times 0.0226\rho U^2\left[\frac{\mu}{\rho U\delta}\right]^{\frac{1}{4}} \qquad (4.3)$$

i.e

$$\frac{d}{dx}\int_o^{\delta}\left[U^2\left(\frac{y}{\delta}\right)^{\frac{1}{7}} - U^2\left(\frac{y}{\delta}\right)^{\frac{2}{7}}\right]dy = 0.0226\ U^2\left[\frac{\mu}{\rho\, U\delta}\right]^{\frac{1}{4}} \qquad (4.4)$$

i.e

$$\frac{d}{dx}\int_o^{\delta}\left[\left(\frac{y}{\delta}\right)^{\frac{1}{7}} - \left(\frac{y}{\delta}\right)^{\frac{2}{7}}\right]dy = 0.0226\left[\frac{\mu}{\rho\, U}\right]^{\frac{1}{4}}\delta^{-\frac{1}{4}} \qquad (4.5)$$

Integrating the lhs of (4.5) and equating with the rhs, we find

$$\frac{d}{dx}\left[\frac{7}{8}y^{8/7}\ \delta^{-1/7} - \frac{7}{9}y^{\frac{9}{7}}\delta^{-\frac{2}{7}}\right]_{y=0}^{y=\delta} = 0.0226\left[\frac{\mu}{\rho\, U}\right]^{\frac{1}{4}}\delta^{-\frac{1}{4}} \qquad (4.6)$$

Simplification of (4.6) yields

$$\frac{7}{72}\frac{d}{dx}\delta = 0.0226\left[\frac{\mu}{\rho\, U}\right]^{\frac{1}{4}}\delta^{-1/4} \qquad (4.7)$$

i.e

$$\delta^{\frac{1}{4}}d\delta = 0.2324\left[\frac{\mu}{\rho\, U}\right]^{\frac{1}{4}}dx \qquad (4.8)$$

which on integration gives

$$\delta^{5/4} = 0.2905\left[\frac{\mu}{\rho\, U}\right]^{\frac{1}{4}}x + C \qquad (4.9)$$

Here C is the constant of integration.

When $x=0, \delta=0\ \ and\ \ C=0,$ so that (4.9) becomes

$$\delta^{5/4} = 0.2905\left[\frac{\mu}{\rho\, U}\right]^{\frac{1}{4}}x \qquad (4.10)$$

i.e

$$\delta = 0.3719\left[\frac{\mu}{\rho\, U}\right]^{\frac{1}{5}} x^{\frac{4}{5}} \qquad (4.11)$$

i.e

$$\delta = 0.3719\left[\frac{\mu}{\rho\, Ux}\right]^{\frac{1}{5}} x \qquad (4.12)$$

or

$$\frac{\delta}{x} = \frac{0.3719}{(\mathrm{Re}_x)^{1/5}} \qquad (4.13)$$

(ii) Wall shear stress τ_{approx}

Substituting (4.13) into (2.11) we find

$$\tau_{approx} = 0.0226\rho\, U^2\left[\frac{\mu}{\rho\, U.\dfrac{0.3719x}{(\mathrm{Re}_x)^{1/5}}}\right]^{\frac{1}{4}} \qquad (4.14)$$

This simplifies to

$$\tau_{approx} = 0.0289\rho\, U^2\left[\frac{(\mathrm{Re}_x)^{\frac{1}{5}}}{\mathrm{Re}_x}\right]^{\frac{1}{4}} \qquad (4.15)$$

or

$$\tau_{approx} = 0.0289\rho\, U^2\ (\mathrm{Re}_x)^{-\frac{1}{5}} \qquad (4.16)$$

(iii) Skin friction coefficient C_f

This is given by Schlichting [9]

$$C_f = \frac{\tau_{approx}}{\frac{1}{2}\rho\, U^2} \qquad (4.17)$$

Substituting (4.16) into (4.17) and simplifying we obtain

$$C_f = \frac{0.0576}{(\mathrm{Re}_x)^{1/5}} \qquad (4.18)$$

5.0 Illustrative Examples

5.1 Case of laminar boundary layer flow

Consider selected values of Reynolds numbers (Re) typical of laminar boundary layer flow, viz: 5×10^4, 6×10^4, 7×10^4, 8×10^4, 9×10^4. Using these in the relations (2.6), (2.7), (3.17) and (3.23), the exact and approximate values of the two characteristics in laminar zone are determined and compared. The result is displayed in Table 1.

Table 1: Reynolds Numbers with Exact and Approximate Values of Two Characteristics in Laminar Boundary Layer Flow

Reynoldsk Number (Re)	Exact Blasiue values		Approximate values	
	$\frac{\delta_{Blaius}}{x} = \frac{5}{(\mathrm{Re}_x)^{1/2}}$	$C_f = \frac{0.664}{(\mathrm{Re}_x)^{1/2}}$	$\frac{\delta_{approx}}{x} = \frac{4}{(\mathrm{Re}_x)^{1/2}}$	$C_f = \frac{0.5}{(\mathrm{Re}_x)^{1/2}}$
5×10^4	2.23×10^{-2}	2.96×10^{-3}	1.78 x 10^{-2}	2.23 x 10^{-3}
6×10^4	2.04×10^{-2}	2.71×10^{-3}	1.63 x 10^{-2}	2.04 x 10^{-3}
7×10^4	1.88×10^{-2}	2.50×10^{-3}	1.51 x 10^{-2}	1.88 x 10^{-3}
8×10^4	1.76×10^{-2}	2.34×10^{-3}	1.41 x 10^{-2}	1.76 x 10^{-3}
9×10^4	1.66×10^{-2}	2.21×10^{-3}	1.33 x 10^{-2}	1.66 x 10^{-3}

5.2 Case of turbulent boundary layer flow

Similarly, we consider values of Reynolds numbers in the turbulent boundary layer flow selected arbitrarily, namely: 5x10^6, 6x10^6, 7x10^6, 8x10^6 and 9x10^6. Again, using these in the model (2.8), (2.10), (4.13) and (4.18), the exact and approximate values of the two characteristics in the turbulent zone are also determined and compared. The result is displayed in Table 2.

Table 2: Reynolds Numbers with Exact and Approximate Values of Two Characteristics in Turbulent Boundary Layer Flow.

Reynolds Number (Re)	Exact Blasiue values		Approximate values	
	$\frac{\delta_{Blaius}}{x} = \frac{0.382}{(\mathrm{Re}_x)^{1/5}}$	$C_f = \frac{0.0574}{(\mathrm{Re}_x)^{1/5}}$	$\frac{\delta_{approx}}{x} = \frac{0.3719}{(\mathrm{Re}_x)^{1/5}}$	$C_f = \frac{0.0578}{(\mathrm{Re}_x)^{1/5}}$
5×10^6	1.74×10^{-2}	2.62×10^{-3}	1.70 x 10^{-2}	2.64 x 10^{-3}
6×10^6	1.68×10^{-2}	2.53×10^{-3}	1.63 x 10^{-2}	2.54 x 10^{-3}
7×10^6	1.63×10^{-2}	2.45×10^{-3}	1.59 x 10^{-2}	2.47 x 10^{-3}
8×10^6	1.59×10^{-2}	2.38×10^{-3}	1.54 x 10^{-2}	2.40 x 10^{-3}
9×10^6	1.55×10^{-2}	2.33×10^{-3}	1.51 x 10^{-2}	2.35 x 10^{-3}

Discussion and conclusion

From Tables 1 and 2, we observe that the error, corrected to two places of decimals, in the boundary layer thickness in respect of the laminar flow is greater than the error in the boundary layer thickness in respect of the turbulent flow. Thus, the boundary layer thickness in the turbulent zone has a higher accuracy index than its counterpart in the laminar zone.
The trend in the case of the second characteristic, namely, skin friction coefficient, is the same. That is, from the two Tables, the error in the skin friction coefficient in the laminar layer is also greater than the error in the skin friction coefficient in the turbulent layer. Hence, the skin friction coefficient in the turbulent flow also has a higher accuracy index than its counterpart in the laminar flow. This behavior is attributable to the fact that the boundary layer thickness and the skin friction coefficient in the turbulent zone grow more rapidly than their counterparts in the laminar zone because the exponent of Re_x in the turbulent layer is $\frac{1}{5}$ compared to the exponent of Re_x in the laminar layer which is but $\frac{1}{2}$.

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