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An Error Estimation of a numerical scheme analogous to the Tau method for initial value problems in Ordinary Differential Equations

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Abstract

In a recent paper, we constructed three classes of orthogonal polynomials for use in the perturbation term of a numerical integration scheme analogous to the tau method of Lanczos and Ortiz for ordinary differential equations. The resulting $n-th$ degree approximant, $y_n(x)$ of the solution $y(x)$ of the differential equation was accurate and hence justi_ed the scheme. In this present paper, we report an error estimation of the method based on our earlier work. The estimate obtained is good as it correctly captures the order of the tau approximant.

1.0 Introduction

The tau method of Lanczos solves the $m-th$ order differential equation:

$$Ly(x)=\sum_{r=0}^{m}\sum_{k=0}^{N_r}\left(P_{rk}x^k\right)y^{(r)}(x)=\sum_{r=0}^{\sigma}f_rx^r,a\le x\le b \tag{1.1a}$$

$$L^*y(x_{rk})\equiv\sum_{r=0}^{m-1}a_{rk}y^{(r)}(x_{rk})=\alpha_r,k=1(1)m \tag{1.1b}$$

by seeking an approximant

$$y_n(x)=\sum_{r=0}^{n}a_rx^r,r<+\infty \tag{1.2}$$

of $y(x)$ which is the exact solution of the corresponding perturbed system

$$Ly_n(x)=\sum_{r=0}^{\sigma}f_rx^r+H_n(x) \tag{1.3a}$$

$$L^*y_n(x_{rk})=\alpha_k,k=1(1)m \tag{1.3b}$$

where $\propto_k, f_k, P_{r,k}, N_r; r=0(1)m, k=0(1)N_r$ are real integers, $y^{(r)}$ denotes the derivates of order r of $y(x)$, the perturbation term $H_n(x)$ in (1.3a) is defined by:

$$H_n(x)=\sum_{i=0}^{m+s-1}\tau_{i+1}T_{n-m+i+1}(x)=\sum_{i=0}^{m+s-1}\tau_{i+1}\sum_{r=0}^{n-m+i+1}C_r^{(n-m+i+1)}x^r \tag{1.4}$$

and $C_r^{(n)}$ is the coefficient of x^r in the $n-th$ degree chebyshev polynomial $T_n(x)$; that is,

$$T_n(x)=\cos\left(n\cos^{-1}\left\{\frac{2x-a-b}{b-a}\right\}\right)=\sum_{r=0}^{n}C_r^{(n)}x^r \tag{1.5}$$

The $\tau's$ are free parameters to be determined and s, the number of overdetermination of (1.1a), is defined by:

$$s=\max\left\{N_r-r>0\middle|0\le r\le m\right\}$$

For different order m and s, (that is $m=1,2,\ldots$ and $s=1,2,\ldots$).
In Issa and Adeniyi [1], we replaced (1.1) by certain orthogonal polynomial and showed that the resulting approximant $y_n(x)$ of $y(x)$ was accurate and favourably compared with the approximant obtained from (1.2).

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In the next section we shall briefly review an error estimation of Adeniyi et al [2, 3] for the tau method (1.3) and construction of orthogonal polynomials (Nu-Polynomials). In section 3 we shall follow Adeniyi et al [3, 4] to estimate the error of the method reported in [1]. Section 4 focuses on experimentation with the error estimate. The paper is finally concluded in section 5 with some concluding remarks.

2.0 Review of Orthogonal Polynomials (Nu-Polynomials)

Orthogonal polynomials have been widely used in problems involving the approximation of functions such as in the economization of power series, mini-max approximation, Gaussian quadrature techniques, solution of both integral and differential equation as well as in collocation techniques, among others [2, 3, 5, 6-12].

The construction of these polynomials may be based on the three equations of Gram Schmidt orthogonalization principle namely:

$$\varphi_n(x)=\sum_{r=0}^{n}K_r^{(n)}x^r \tag{2.1a}$$

$$\varphi_n(1)=1 \tag{2.1b}$$

$$\left\langle\varphi_m(x),\varphi_n(x)\right\rangle=0\equiv\int_a^b w(x)\varphi_m(x)\varphi_n(x)dx, m\neq n \tag{2.1c}$$

Where $\varphi_m(x), a\leq x\leq b$ is the orthogonal polynomial in question.

By employing equations (2.1) for three cases of $w(x)$ over $a\leq x\leq b$, we obtained the following results:

Case 1: $w(x)=1+x,\ 0\leq x\leq 1$, we have:

$\varphi_0(x)=1, \varphi_1(x)=-2+3x, \varphi_2(x)=3-12x+10x^2$ and so on

Case 2: $w(x)=1-x,\ 0\leq x\leq 1$, we have:

$\varphi_0(x)=1, \varphi_1(x)=\frac{1}{2}(-1+3x), \varphi_2(x)=\frac{1}{3}(1-8x+10x^2)$ and so on

Case 3: $w(x)=1-x^2,\ 0\leq x\leq 1$, we have:

$\varphi_0(x)=1, \varphi_1(x)=-1+2x\ , \varphi_2(x)=1-5x+5x^2$ and so on

2.1 An Error Estimation of the Tau Method

Based on the error of the Lanczos economization process, Adeniyi et al [2, 3, 4] constructed an error polynomial:

$$\left(e_n(x)\right)_{n+1}=\frac{\phi_n(x-\alpha)^m T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}}, a\leq x\leq b \tag{2.1.1}$$

as an approximation to the error

$$e_n(x)=y(x)-y_n(x) \tag{2.1.2}$$

in $y(x)$, obtained from the Tau approximation process (1.3). The parameter $\emptyset_n$ is to be determined along with other parameters defined in the next section while $a\leq x\leq b$ is some point at which the conditions (1.3b) is specified. The details of the procedure for the determination of $\left(e_{n(x)}\right)_{n+1}$ is presented in the next section.

3.0 An Error estimation of an analogue of the Tau Method for ODEs

Following Adeniyi et al [2], we defined error function (2.1.2) as $e_n(x)=y(x)-y_n(x)$ which is the exact solution of the error differential system:

$$Le_n(x)=-\sum_{i=0}^{m+s-1}\nu_{i+1}\varphi_{n-m+i+1}(x) \tag{3.1a}$$

$$L^{*}e_n(x)=0 \tag{3.1b}$$

where ν_r replaces the τ_r in (1.3a).

Now, closely similar to the error polynomial (2.1.1) is the error approximant

$$\left(e_n(x)\right)_{n+1}=\frac{\lambda_n(x-\alpha)^m\varphi_{n-m+1}(x)}{K_{n-m+1}^{(n-m+1)}}, a\leq x\leq b \tag{3.2}$$

of $e_n(x)$ in (2.1.2) and which satisfies exactly the perturbed error differential system:

$$\left(e_n(x)\right)_{n+1} = -\sum_{i=0}^{m+s-1} \nu_{i+1}\varphi_{n-m+i+1}(x) + \sum_{i=0}^{m+s-1} \gamma_{i+1}\varphi_{n-m+i+2}(x) \tag{3.3a}$$

$$L^*\left(e_n(\alpha)\right)_{n+1} = 0 \tag{3.3b}$$

The extra parameters $\gamma_1, \gamma_2, \ldots, \gamma_{m+s}$ in (3.3a) are to be determined along with λ_n in (3.2).
Equating the corresponding coefficients of $x^{n+s+1}, x^{n+s}, \ldots, x^{n-m+1}$ in (3.3) gives $m+s$ equations for the unique determination of $\gamma_1, \gamma_2, \ldots, \gamma_{m+s}$ and λ_n, a forward elimination process is recommended for the solution of this resulting linear system. The value of λ_n is then inserted into (3.2) and subsequently, we get the estimate

$$\xi_n = \max\left|\left(e_n(\alpha)\right)_{n+1}\right| : a \le x \le b = \left|\frac{\lambda_n}{C_{n-m+1}^{(n-m+1)}}\right| \tag{3.4}$$

of the maximum error

$$\xi_n^* = \max\left|e_n(x)\right|, a \le x \le b \tag{3.5}$$

4.0 Numerical Examples

We consider three problems for implementation with the method in the above discourse

Example 4.1

$$y''(x) + y(x) = 0,\ y(0) = 1, y'(0) = 0,\ 0 \le x \le 1$$

With analytical solution $y(x) = \cos(x)$

Using (1.3a), we have

$$Ly_n(x) = \nu_1\varphi_{n-1}(x) + \nu_2\varphi_n(x) \tag{4.1}$$

where $y_n(x)$ is given by (1.2), ν_1 and ν_2 are to be determined, $\varphi_n(x)$ can be any of the orthogonal polynomials discoursed in section 2

From (4.1) and the given problem, we have

$$Le_n(x) = -\nu_1\varphi_{n-1}(x) - \nu_2\varphi_n(x)$$

where $e_n(x)$ is given in (2.1.2)

Seeking an approximant (3.2), we have

$$E_n(x) = \frac{\lambda_n x^2 \varphi_{n-1}(x)}{K_{n-1}^{(n-1)}},\ (m = 2, \alpha = 0) \tag{4.2}$$

Again, substituting (4.2) in (3.3) we obtain

$$L\left(E_n(x)\right) = -\nu_1\varphi_{n-1}(x) - \nu_2\varphi_n(x) + \gamma_1\varphi_n(x) + \gamma_2\varphi_{n+1}(x) \tag{4.3}$$

where

$$L = \frac{d^2}{dx^2} + 1$$

Equating the corresponding coefficient in (4.3), and solve the resulting equation for different cases of $\varphi_r(x), r = 0, 1, \ldots n$.
See Table 4.1 for the numerical results of the error estimate.

Example 4.2

$$2(1+x)y'(x) + y(x) = 0,\ \ y(0) = 1,\ 0 \le x \le 1$$

With analytical solution $y(x) = \left(1+x\right)^{-\frac{1}{2}}$

Numerical results for the error estimate for this example are presented in Table 4.2.

Example 4.3

$$y'(x) - x^2 y(x) = 0,\ \ y(0) = 1,\ 0 \le x \le 1$$

With analytical solution $y(x) = e^{\frac{x^3}{3}}$

Numerical results for the error estimate for this example are presented in Table 4.3.

Table 4.1: Error Estimate for Example 4.1

Weight function ($w(x)$)	Degree 5 (ξ_5)	Degree 6 (ξ_6)	Degree 7 (ξ_7)
$1+x$	9.16×10^{-6}	1.78×10^{-7}	1.03×10^{-8}
$1-x$	4.82×10^{-6}	1.12×10^{-6}	8.88×10^{-8}
$1-x^2$	2.83×10^{-5}	7.06×10^{-7}	5.00×10^{-8}
Chebyshev	1.34×10^{-5}	1.81×10^{-7}	1.04×10^{-8}

Table 4.2: Error Estimate for Example 4.2

Weight function ($w(x)$)	Degree 5 (ξ_5)	Degree 6 (ξ_6)	Degree 7 (ξ_7)
$1+x$	2.55×10^{-5}	4.38×10^{-6}	7.31×10^{-7}
$1-x$	2.26×10^{-4}	4.49×10^{-5}	1.00×10^{-5}
$1-x^2$	1.07×10^{-4}	2.09×10^{-5}	4.02×10^{-6}
Chebyshev	2.90×10^{-5}	4.60×10^{-6}	7.37×10^{-7}

Table 4.3: Error Estimate for Example 4.3

Weight function ($w(x)$)	Degree 5 (ξ_5)	Degree 6 (ξ_6)	Degree 7 (ξ_7)
$1+x$	4.35×10^{-4}	4.14×10^{-5}	4.83×10^{-6}
$1-x$	6.92×10^{-3}	5.61×10^{-4}	4.96×10^{-5}
$1-x^2$	1.33×10^{-3}	2.52×10^{-4}	2.58×10^{-5}
Chebyshev	4.94×10^{-4}	4.42×10^{-5}	2.58×10^{-5}

5.0 Conclusion

An error estimation of an analogue of the tau method for initial value problems in ordinary differential equations has been presented. Results for three variants based on three classes of orthogonal polynomials show that the error estimate is good, especially $w(x) = 1 + x$ which give more accurate result compare to that of chebyshev polynomial and the others also captured the Tau approximant correctly.

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