Journal of the Nigerian Association of Mathematical Physics
Volume 26, (*March, 2014*), pp 73 – 77

Modified Third Order Predictor-Corrector Method for Solving Systems of Nonlinear Equations

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Abstract

In this paper, we have proposed a modified iterative method suggested by Ogbereyivwe and Emunefe for the solution of non linear equations to solving system of nonlinear equations. The proposed method has been illustrated with several examples from the references. The numerical results indicate that this proposed method provide good performance of iterations.

Keywords: Systems of Nonlinear equation, Newton's method, Computational order of Convergence.

1.0 Introduction

Systems of Nonlinear phenomena play a crucial role in applied mathematics and physics. The solutions of systems of algebraic equations have a well-developed mathematical and computational theory. The situation is much more complicated when the equations in the system do not exhibit nice linear or polynomial properties. In this case, both the mathematical theory and computational practices are far from complete understanding of the solution process. Thus, we are left with numerical routines to determine the roots of the system.

Recently, several iterative methods have been used to solve nonlinear equations and several authors have modified some of these methods to solve systems of nonlinear equations [1-7].

The purpose of this study is to introduce an extension of Ogbereyivwe and Emunefe [8] method for solving nonlinear equations to solving systems of nonlinear equations. Some examples are tested and obtained results suggest that this newly improvement technique introduces a promising tool and powerful improvement for solving a System of Nonlinear Equations.

2.0 Development of the Method

We begin by setting the notation. We seek to solve the systems of nonlinear equation

$$F(x) = 0 \qquad (1)$$

Here $F: R^N \to R^N$. We denote the ith component of F by f_i. If the component of F are differentiable at $x \in R^N$ we define the *Jacobian matrix* $F^{'}(x)$ by

$$F^{'}(x)_{ij} = \frac{\partial f_i}{\partial x_j}(x) \qquad (2)$$

where the Jacobian matrix is the vector analog of the derivative [9]. Consider the fundamental theorem of calculus as follows.

Theorem 1

Let F be differentiable in an open set $\Omega \subset R^N$ and let $X_i \in \Omega$. Then for all $X \in \Omega$ sufficiently near X_i

$$F(X) - F(X_i) = \int_{X_i}^{X} F^{'}(t)\, dt \qquad (3)$$

If we approximate the integral in (3) by Composite Trapezoidal rule given by

$$\int_{x_i}^{x} F^{'}(t)\, dt = \frac{X - X_i}{4}\left[F^{'}(X_i) + 2F^{'}\left(\frac{X_i + X}{2}\right) + F^{'}(X)\right] \qquad (4)$$

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from (3) we have

$$F(X) = F(X_i) + \frac{X - X_i}{4}\left[F'(X_i) + 2F'\left(\frac{X_i + X}{2}\right) + F'(X)\right] \quad (5)$$

Since $F(X) = 0$ then

$$X = X_i - 4\left[F'(X_i) + 2F'\left(\frac{X_i + X}{2}\right) + F'(X)\right]^{-1} F(X_i) \quad (6)$$

which is an implicit method. To overcome this setback, we use the predictor-correction method. If we chose the Newton's method as the predictor defined as

$$X_+ = X_i - F'(X_i)^{-1}F(X_i) \quad (7)$$

Where the iteration transition is from X_i to X_+. Hence we suggest the following algorithm.

Algorithm 1: For a given X_0, compute the approximate solution X_+ by the iterative scheme

$$X_+ = X_i - F'(X_i)^{-1}F(X_i) \quad (8)$$

$$X = X_i - 4\left[F'(X_i) + 2F'\left(\frac{X_i + X_+}{2}\right) + F'(X_+)\right]^{-1} F(X_i) \quad (9)$$

Algorithm 1 can be considered as a modified [8] proposed iterative method for solving nonlinear equations stated as:
Given an initial approximation x_0 (close to α the root of $f(x) = 0$, we find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 2f'\left(\frac{x_n + y_n}{2}\right) + f'(y_n)}, \quad n = 0,1,2, \ldots \quad (10)$$

The convergence analysis and its applicability are presented in [8].

3.0 Convergence analysis

In this section we consider the convergence of our method. We achieve this using the Taylor's series method.

Theorem 1: Let X^* be a simple zero of sufficient differentiable function $F: R^N \to R^N$ for an open interval. If X_0 is sufficiently close to X^*, then the two step method defined by our algorithm (1) has convergence at least of order 3.

Proof Let X^* be a simple zero of F, and $E_n = X_n - X^*$. Using Taylor expansion around $X = X^*$ and taking into account $F(X^*) = 0$, we get

$$F(X_n) = F'(X^*)[E_n + C_2E_n^2 + C_3E_n^3 + C_4E_n^4 + O(\|E_n^4\|) \ldots] \quad (11)$$

$$F'(X_n) = F'(X^*)[I + 2C_2E_n + 3C_3E_n^2 + 4C_4E_n^3 + O(\|E_n^4\|) \ldots] \quad (12)$$

Where

$$C_k = \frac{F^k(X^*)}{k!\,F'(X^*)}, \quad k = 2,3,4, \ldots \quad (13)$$

Using (11) and (12), we have;

$$X_+ = X_i - F'(X_i)^{-1}F(X_i)$$

$$= \left[X^* + C_2E_n^2 + 2({C_2}^2 - C_3)E_n^3 - (7C_2C_3 - 4C_2^3 - 3C_4)E_n^4 + O(\|E_n^5\|) \ldots\right] \quad (14)$$

$$F'(X_+) = F'(X^*)\left[I + 2{C_2}^2E_n^2 + 4\left(C_2C_3 - {C_2}^3\right)E_n^3 + (-11{C_2}^2C_3 + 6C_2C_4 + 8C_2^4)E_n^4 + O(\|E_n^5\|) \ldots\right] \quad (15)$$

and

$$F'\left(\frac{X_n + X_+}{2}\right) = F(X^*)\left(I + C_2E_n + \left(C_2^2 + \frac{3}{4}C_3\right)E_n^2 + \left(-2C_2^3 + \frac{7}{2}C_2C_3 + \frac{1}{2}C_4\right)E_n^3 + \left(\frac{9}{2}C_2C_4 + C_2^4 - \frac{37}{4}C_2^2C_3 + 3C_3^2 + \frac{5}{16}C_5\right)E_n^4 + O(\|E_n^5\|) \ldots \quad (16)$$

Combining (11),(12),(16) and (15) in (9) gives;

$$E_{n+1} = X_i - 4\left[F'(X_i) + 2F'\left(\frac{X_i + X_+}{2}\right) + F'(X_+)\right]^{-1} F(X_i)$$

$$= X^* + \left(\frac{1}{8}C_3 + C_2^2\right)e_n^3 + O(\|E_n^4\|) \quad \blacksquare \qquad (17)$$

This completes the proof. Hence the method described by algorithm 1 is of order 3.

4.0 Numerical examples

In order to demonstrate the performance of the introduced iterative method (Algorithm 1) as a solver for systems of nonlinear equations, several different problems were selected as test problems. We present the results of our comparison of the method (OM) with the classical Newton's method (NM). The calculations were done using MATLAB. Our comparison of the methods is based upon the following criteria: number of iterations and the Computational order of Convergence. We use the following stopping criterion for our computer programs:

$$\|X^{n+1} - X^n\| + \left\|F\left(X^{(n)}\right)\right\| < 10^{-15}$$

The computational order of convergence $\rho(i,k)$, for series $\left\{x_k^{(i)}\right\}$, is computed by:

$$\rho(i,k) = \frac{\ln\left|\left(x_{k+1}^{(i)} - \alpha\right)/\left(x_k^{(i)} - \alpha\right)\right|}{\ln\left|\left(x_k^{(i)} - \alpha\right)/\left(x_{k-1}^{(i)} - \alpha\right)\right|}, i = 1,2,\dots,n, \quad k \geq 2 \quad (see[9])$$

where α is the solution of the system.

We introduce the following notations: x_0: an initial approximation, *Iter*: number of iterations, *COC* : computational order of convergence.

Example 1

Consider the following system of nonlinear equations [10]:

$$F_1(X) = \begin{cases} f_1(x,y) = x^2 + xy + y^2 - 7 = 0 \\ f_2(x,y) = x^3 + y^3 - 9 = 0 \end{cases} \qquad X_0 = (1.5,0.5)$$

Example 2

Consider the following system of nonlinear equations [6], [11]:

$$F_2(X) = \begin{cases} f_1(x,y) = x^2 - 10x + y^2 + 8 = 0 \\ f_2(x,y) = xy^2 + x - 10y + 8 = 0 \end{cases} \qquad X_0 = (2,2)$$

Example 3

Consider the following system of nonlinear equations [6], [11]:

$$F_3(X) = \begin{cases} f_1(x,y,z) = 15x + y^2 - 4z - 13 = 0 \\ f_2(x,y,z) = x^2 + 10y - e^{-z} - 11 = 0 \\ f_3(x,y,z) = y^3 - 25z + 22 = 0 \end{cases}, \quad X_0 = (10,6,-5)$$

Example 4

Consider the following system of nonlinear equations [10]:

$$F_4(X) = \begin{cases} f_1(x,y,z) = 10x + \sin(x+1) - 1 = 0 \\ f_2(x,y,z) = 8y - cos^2(z-y) - 1, \\ f_3(x,y,z) = 12z + sinz - 1 = 0 \end{cases} \qquad X_0 = \left(\frac{1}{10}, \frac{1}{4}, \frac{1}{12}\right)$$

Example 5
Consider the following system of nonlinear equations [6], [11]:

$$F_5(X) = \begin{cases} f_1(x,y,z) = 3x - \cos(yz) - 0.5 = 0 \\ f_2(x,y,z) = x^2 - 81(y+0.1)^2 + \sin z + 1.06 = 0\,, \quad X_0 = (1.1,1.1,1.1) \\ f_3(x,y,z) = e^{-xy} + 20z + \dfrac{10\pi - 3}{3} = 0 \end{cases}$$

Table 1 shows the number of iterations and the computational order of convergence for the proposed method (OM) and the Newton's method (NM).

Table 1: Number of iterations and Computational order of Convergence for Examples 1-5

Functions and Methods	***Iter***	***COC***
$F_1, X_0 = (1.5,0.5)$		
NM	6	1.99856990615671
OM	4	3.16358842584268
$F_2, X_0 = (2,2)$		
NM	8	2.00467083335282
OM	5	2.84569066468143
$F_3, X_0 = (10,6,-5)$		
NM	8	2.26089072336615
OM	5	3.23147391984341
$F_4, X_0 = \left(\frac{1}{10}, \frac{1}{4}, \frac{1}{12}\right)$		
NM	3	1.91397104283141
OM	2	3.27604987909553
$F_5, X_0 = (1.1,1.1,1.1)$		
NM	8	2.00043456732353
OM	5	3.00000000061232

In examples below, we tested the proposed method on spares systems with m unknown variables. We also compare the performance of the proposed OM method with that of NR method focusing on iteration number. We take $\varepsilon = 10^{-15}$ as tolerance.

Example 6
Consider the following system of nonlinear equations [6], [11]:

$$F_6: f_i = e^{x_i} - 1 \quad , i = 1,2,\dots,m. \quad X_0 = [0.5, 0.5, \dots, 0.5]^T$$

The results are presented in **Table 2.**

Example 7
Consider the following system of nonlinear equations [12]:

$$F_7: f_i = x_i^2 - \cos(x_i - 1) = 0, \quad i = 1,2,\dots,m$$

To solve this system, we set $X_0 = [0.5, 0.5, \dots, 0.5]^T$ as an initial value. The results are presented in **Table 2**.

Table 2: Number of iterations for Example 6 and 7

Methods	F_5	F_6	F_5	F_6	F_5	F_6
	$m = 7$		$m = 50$		$m = 100$	
NM	6	7	6	7	6	7
OM	4	5	4	5	4	5

m – number of equations in the system in Example 6 and 7

5.0 Conclusion

In this paper, we have proposed a modified method for solving system of nonlinear equations. We have also demonstrated the applicability of the modified method on some concrete examples and its performance was compared with that of classical Newton's method. Our proposed method provides highly accurate results in a less number of iterations as compared with Newton's method.

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