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Third derivative GLM with RK-stability

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Abstract

This paper presents third derivative Runge-Kutta methods (TDRK) which have a simple transformation to general linear method (GLM) for the numerical integration ofinitial value problems (IVPs) in ordinary differential equations (ODEs).

Keywords: *Rung-Kutta methods; third derivative RK methods, GLM, Explict RK method.*
AMS subject classification: 65L05, 65L06.

1.0 Introduction

The RK method is one of the traditional methods for the numerical solution of IVPs in ODEs,

$$\begin{cases} y' = f(x, y(x)), x \in [x_0, X], \\ y_0 = y(x_0), \quad y \in R^m, \ f \in R \times R^m, m \geq 1. \end{cases} \tag{1}$$

The RK method is

$$Y_i = y_{n-1} + h\sum_{j=1}^{s} a_{ij} f(Y_j), \quad i = 1, \ldots, s, \tag{2}$$

$$y_n = y_{n-1} + h\sum_{i=1}^{s} b_i f(Y_i), \tag{3}$$

where, $h = x_n - x_{n-1}$ is the step length and the stage $Y_i = y(x_{n-1} + c_i h) + 0(h^{q+1})$, the $c_i = [c_1, c_2, \ldots, c_s]^T$ is called the abscissa vector or the *nodes* and it may lie between 0 and 1, while q represent the stage order. The $y_n = y(x_{n-1} + h) + 0(h^{p+1})$ is denotes the output method and is of order *p*. To specify a particular method, one needs to provide the integer *s* (the number of stages), and the coefficients a_{ij} (for $1 \leq j < i \leq s$), b_i (for $i = 1, 2, ..., s$) and c_i (for $i = 2, 3, ..., s$).With the paper of Butcher [1] it became customary to symbolize methods (1.2) and (1.3) by the tableau

$$\begin{array}{c|c} c_i & [a_{ij}] \\ \hline & b_i^T \end{array}$$

where the matrix $[a_{ij}]$ is called the RK matrix, while the b_i are known as theweights.Examples of RK methods are in [1-7].As in [7], RK methods have both advantages and disadvantages. They are stable and easy to implement in variable step size and order. However, they have difficulties in achieving high accuracy at reasonable computational cost. The interest is on the transformation of RK methods to GLM [8]. The GLM was introduced by Butcher [8] to provide a unifying framework for both multistage and multivalue methods. The GLM[8] is

$$\begin{cases} Y_i = \sum_{j=1}^{s} h\, a_{ij} F_j + \sum_{j=1}^{r} u_{ij} y_j^{[n-1]}, \quad i = 1, \ldots, s, \\ y_i^{[n]} = \sum_{j=1}^{s} h\, b_{ij} F_j + \sum_{j=1}^{r} v_{ij} y_j^{[n-1]}, \quad i = 1, \ldots, r. \end{cases} \tag{4}$$

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The matrix representationis

$$\begin{pmatrix} Y \\ y^{[n]} \end{pmatrix} = \begin{pmatrix} A & U \\ B & V \end{pmatrix} \begin{pmatrix} hF \\ y^{[n-1]} \end{pmatrix}, \qquad (5)$$

where $Y = (Y_1, Y_2, \cdots, Y_S)^T$, $F = (F_1, F_2, \cdots, F_S)^T$ are the stages and stage derivatives while, $y^{[n-1]} = \left(y_1^{[n-1]}, y_2^{[n-1]}, \cdots, y_r^{[n-1]}\right)^T$, $y^{[n]} = \left(y_1^{[n]}, y_2^{[n]}, \cdots, y_r^{[n]}\right)^T$ denotes the input and output approximations. Thematrices in (5) are $A = \{a_{ij}\} \in R^{(s\times r)}$, $B = \{b_{ij}\} \in R^{(r\times r)}$, $U = \{u_{ij}\} \in R^{(s\times s)}$, and $U = \{v_{ij}\} \in R^{(r\times s)}$.The stability of the method in (5) is determined from the stability matrix

$$M(z) = V + zB(I - zA). \qquad (6)$$

The characteristics polynomial of (5) is

$$\Pi(w,z) = \det\left(wI - M(z)\right). \qquad (7)$$

Definition1.c.f. [9]: If the characteristic polynomial of *M(z)*, known as the stability function, has the special form

$$\Pi(w,z) = \det\left(wI - M(z)\right) = w^{r-1}\left(w - R(z)\right),$$

then the method is said to possess RK stability.

Examples ofRK methods in the format of (5) are in [4-5].How this isdone has been well discussed in [10-14]. Examples of RK methods in GLM form are in [10-14].Enright [15] considered second derivative in linear multistep methods. In this regard,Butcher and Hojjati [10], and Okuonghae [12-13] extended ((2), (3)) and (5) to SDGLM methods.An example is

$$\begin{pmatrix} Y \\ y^{[n]} \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & U \\ B_1 & B_2 & V \end{pmatrix} \begin{pmatrix} hF \\ h^2 F' \\ y^{[n-1]} \end{pmatrix}, c = \begin{pmatrix} 0 \\ \frac{2}{3} \\ 1 \end{pmatrix}, \qquad (8)$$

where,

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{7}{16} & \frac{9}{16} & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{27} & 0 & 0 \\ \frac{1}{16} & \frac{1}{16} & 0 \end{pmatrix}, B_1 = \begin{pmatrix} \frac{7}{16} & \frac{9}{16} & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} \frac{1}{16} & \frac{1}{16} & 0 \end{pmatrix},$$

$U = (1 \;\; 1 \;\; 1)^T, V = 1,\; Y = (Y_1, Y_2, Y_3)^T, F = (F_1, F_2, F_3)^T,\; F' = (F_1', F_1', F_3')^T.$

The stability polynomial of the explicit SDGLM (8) is

$$\Pi(w,z) = w - 1 - z - \frac{5z^2}{16} - \frac{z^3}{24} - \frac{z^4}{432}.$$

This (8) is thus by [16] nearly ARK-stable. The interval of absolute stability of the algorithm in (8) is (-8, 0).This paper describes an extension of second derivative RK methods [12] to third derivative RK methods (TDRK) which have a simple transformation to GLM.

2.0 Third derivative GLM (TDGLM)

Thegeneral form of the proposed GLM is

$$\begin{cases} Y_i = \sum_{j=1}^{s} h^3 a_{ij}^{(3)}(c_i)F_j'' + h^2 \sum_{j=1}^{s} a_{ij}^{(2)}(c_i)F_j' + h\sum_{j=1}^{s} a_{ij}^{(1)}(c_i)F_j + \sum_{j=1}^{r} u_{ij}(c_i)y_j^{[n-1]}, \quad i = 1,\ldots,s, \\ y_i^{[n]} = \sum_{j=1}^{s} h^3 b_{ij}^{(3)}(t)F_j'' + h^2 \sum_{j=1}^{s} b_{ij}^{(2)}(t)F_j' + h\sum_{j=1}^{s} b_{ij}^{(1)}(t)F_j + \sum_{j=1}^{r} v_{ij}(t)y_j^{[n-1]}, \quad i = 1,\ldots,r, \\ Y_i = y(x_{n-1} + c_i h),\ F_j = F(x_{n-1} + c_i h, Y_i),\ y_1^{[n-1]} = y_{n-1},\ y_1^{[n]} = y(x_{n-1} + th),\ c_s = t = 1. \end{cases} \tag{9}$$

In Butcher tableau this is,

$$\begin{pmatrix} Y \\ y^{[n]} \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 & U \\ B_1 & B_2 & B_3 & V \end{pmatrix} \begin{pmatrix} hF \\ h^2 F' \\ h^3 F'' \\ y^{[n-1]} \end{pmatrix}, c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_s \end{pmatrix}, \tag{10}$$

where,

$Y = (Y_1, Y_2, \cdots, Y_S)^T$, $F = (F_1, F_2, \cdots, F_S)^T$, $F' = (F_1', F_2', \cdots, F_S')^T$, $F'' = (F_1'', F_2'', \cdots, F_S'')^T$, $y^{[n-1]} = (y_1^{[n-1]}, y_2^{[n-1]}, \cdots, y_r^{[n-1]})^T$, $y^{[n]} = (y_1^{[n]}, y_2^{[n]}, \cdots, y_r^{[n]})^T$, $A_1 = \{a_{ij}^{(1)}\} \in R^{(s\times s)}$, $A_2 = \{a_{ij}^{(2)}\} \in R^{(s\times s)}$, $A_3 = \{a_{ij}^{(3)}\} \in R^{(s\times s)}$, $B_1 = \{b_{ij}^{(1)}\} \in R^{(r\times s)}$, $B_2 = \{b_{ij}^{(2)}\} \in R^{(r\times s)}$, $B_3 = \{b_{ij}^{(3)}\} \in R^{(r\times s)}$, $U = \{u_{ij}\} \in R^{(s\times r)}$, and $V = \{v_{ij}\} \in R^{(r\times r)}$.

The dimension of the method in (10) is $(s+r)\times(3s+r)$. The stability matrix of the TDGLM is obtained when (10) is applied to the Dahlquist [17] test scalar problem $y' = \lambda y$, where λ may be a complex number. Following the idea in [12, 13] we obtain

$$\psi(z) = V + z(B_1 + zB_2 + z^2 B_3)(I - zA_1 - z^2 A_2 - z^3 A_3)^{-1} U, \quad z = \lambda h. \tag{11}$$

The stability polynomial of (10) is

$$\Pi(w,z) = \det(wI - \psi(z)). \tag{12}$$

The advantages of TDRK method (10) is that it promotes high order and large region of absolute stability, especially if the methods are of RK stability[18]. Again, they can be used as a starter for a suitable GLM. This paper is organized as follows. Section 3 discusses the derivation of explicit TDRK methods and their transformation to TDGLM. Section 4 discusses some numerical experiment.

3.0 Derivation of the TDRK methods and their transformation to GLM

To derive (9) we use the following polynomial interpolant [17]

$$y(x) = \sum_{j=0}^{N} \theta_j x^j. \tag{13}$$

The variable x in (13) can be computed fromthe scale variable $t = \frac{x - x_{n-1}}{h}$.When $r = 1, s = 1$in(9), we have the first stage and the output methods as

$$\begin{cases} Y_1 = h^3 a_{11}^{(3)}(c_1)F_1'' + h^2 a_{11}^{(2)}(c_1)F_1' + h a_{11}^{(1)}(c_1)F_1 + u_{11}(c_1)y_1^{[n-1]}, \\ y_1^{[n]} = h^3 b_{11}^{(3)}(t)F_1'' + h^2 b_{11}^{(2)}(t)F_1' + h b_{11}^{(1)}(t)F_1 + v_{11}(t)y_1^{[n-1]}, \\ Y_1 = y(x_{n-1} + c_1 h), \quad y_1^{[n]} = Y_2 = y(x_{n-1} + th). \end{cases} \tag{14}$$

The coefficients of the method in (14) are:

$$a_{11}^{(1)}(c_1)=c_1,\ a_{11}^{(2)}(c_1)=\frac{c_1^{\ 2}}{2},\ a_{11}^{(3)}(c_1)=\frac{c_1^{\ 3}}{6},\ u_{11}(c_1)=1,$$

$$b_{11}^{(1)}(t)=t,\ b_{11}^{(2)}(t)=\frac{t^2}{2},\ b_{11}^{(3)}(1)=\frac{t^3}{6},\ v_{11}(t)=1.$$

Fixing $c_1=0$ and $t=1$ gives the stage and the output method

$$Y_1=y_1^{[n-1]}, \tag{15}$$

$$y_1^{[n]}=\frac{h^3}{6}F_1''+\frac{h^2}{2}F_1'+hF_1+y_1^{[n-1]},\quad p=3. \tag{16}$$

The GLM form of (15) and (16) is

$$\begin{pmatrix} Y \\ y^{[n]} \end{pmatrix}=\begin{pmatrix} A_1 & A_2 & A_3 & U \\ B_1 & B_2 & B_3 & V \end{pmatrix}\begin{pmatrix} hF \\ h^2F' \\ h^3F'' \\ y^{[n-1]} \end{pmatrix}, c=\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{17}$$

here,

$$A_1=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, A_2=\begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}, A_3=\begin{pmatrix} 0 & 0 \\ \frac{1}{6} & 0 \end{pmatrix}, B_1=\begin{pmatrix} 1 & 0 \end{pmatrix}, B_2=\begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix}, B_3=\begin{pmatrix} \frac{1}{6} & 0 \end{pmatrix},$$

$$U=\begin{pmatrix} 1 & 1 \end{pmatrix}, V=1,\ Y=(Y_1,Y_2)^T, F=(F_1,F_2)^T, F'=(F_1',F_2')^T, F''=(F_1'',F_2'')^T.$$

The stability polynomial of the TDGLM (17) is $\Pi(w,z)=w-1-z-\frac{z^2}{2}-\frac{z^3}{6}$. The stability polynomial of (17) is exactly the same as that of the classical third order RK method. By implication, this method will behave like the third order RK method. The interval of absolute stability of the method in (17) is (-2.513, 0). The stability plot is given in Fig. 1.

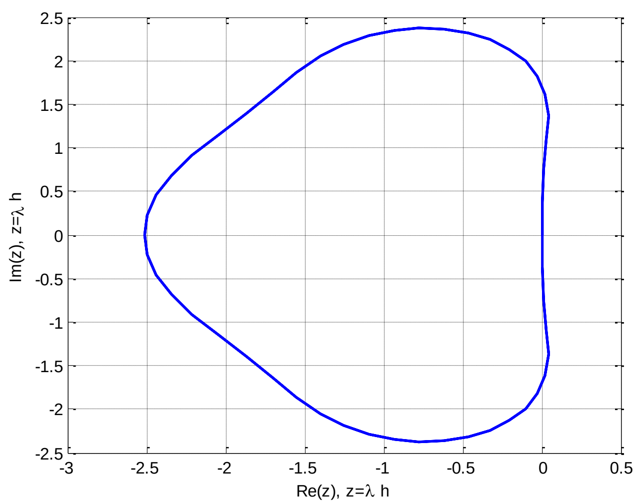

Fig. 1: The stability region of the GLM (17).

Similarly, the coefficient of the methods in (9) when $r=1, s=2$ are

$$a_{11}^{(1)}(c_1)=c_1,\ a_{11}^{(2)}(c_1)=\frac{c_1^{\ 2}}{2},\ a_{11}^{(3)}(c_1)=\frac{c_1^{\ 3}}{6},\ u_{11}(c_1)=1,$$

$$a_{21}^{(1)}(c_2)=c_2,\ a_{21}^{(2)}(c_2)=\frac{c_2^{\ 2}}{2},\ a_{21}^{(3)}(c_2)=\frac{c_2^{\ 3}}{6},\ u_{21}(c_2)=1,$$

$$b^{(1)}{}_{11}=(t(2t^5-6t^4c_2+5t^3c_2^2-2c_2^5+5c_1^2(t-2c_2)(t^2+2c_2(-t+c_2))$$
$$+c_1(-6t^4+10c_2(2t^3-2t^2c_2+c_2^3))))/(2(c_1-c_2)^5),$$
$$b^{(1)}{}_{12}=(-t(2t^5-6t^4c_1+5t^3c_1^2-2c_1^5+2(-3t^4+5c_1(2t^3-2t^2c_1+c_1^3))c_2$$

$$+5(t-2c_1)(t^2+2c_1(-t+c_1))c_2^2))/2\,(c_1-c_2)^5,$$

$$b^{(2)}{}_{11}=(-t(5t^5+2t^3c_1(-7t+5c_1)-2t^2(8t^2+5c_1(-5t+4c_1))c_2)$$
$$+15t(t-2c_1)^2c_2^2+20(t-2c_1)c_1c_2^3-5(t-2c_1)c_2^4))\,/\,10(c_1-c_2)^4,$$

$$b^{(2)}{}_{12}=-(t(5t^5-5c_1^4(t-2c_2)+15tc_1^2(t-2c_2)^2+20c_1^3(t-2c_2)c_2$$
$$+2t^3c_2(-7t+5c_2)-2t^2c_1(8t^2+5c_2(-5t+4c_2)))))/10(c_1-c_2)^4,$$

$$b^{(3)}{}_{11}=(t(10t^5+t^2c_2(-36t^2+5(9t-4c_2)c_2)+15c_1^2(t-2c_2)(t^2+2c_2$$
$$(-t+c_2))+6tc_1(-4t^3+5c_2(3t^2+2c_2(-2t+c_2)))))/(120(c_1-c_2)^3),$$

$$b^{(3)}{}_{12}=(-t(2t^5-6t^4c_1+5t^3c_1^2-2c_1^5+2(-3t^4+5c_1(2t^3-2t^2c_1+c_1^3))c_2$$
$$+5(t-2c_1)(t^2+2c_1(-t+c_1))c_2^2))/2\,(c_1-c_2)^5.$$

Setting $c_1=0$, $c_2=\frac{2}{3}$ and $t=1$ in the above coefficients gives the methods for $r=1, s=2$ in (9) as

$$Y_1=y_1^{[n-1]}\,, \qquad (18)$$

$$Y_2=\frac{4h^3}{81}F_1''+\frac{2h^2}{9}F_1'+\frac{2h}{3}F_1+y_1^{[n-1]}\,, \quad p=3, \qquad (19)$$

$$y_1^{[n]}=h^3\left(\frac{-1}{480}F_1''+\frac{3}{160}F_2''\right)-h^2\left(\frac{1}{160}F_1'+\frac{9}{160}F_2'\right)+h\left(\frac{5}{32}F_1+\frac{27}{32}F_2\right)+y_1^{[n-1]}\,,\ p=6. \qquad (20)$$

The Butcher picture of (18) - (20) is

$$\begin{pmatrix} Y \\ y^{[n]} \end{pmatrix}=\begin{pmatrix} A_1 & A_2 & A_3 & U \\ B_1 & B_2 & B_3 & V \end{pmatrix}\begin{pmatrix} hF \\ h^2F' \\ h^3F'' \\ y^{[n-1]} \end{pmatrix},\ c=\begin{pmatrix} 0 \\ \frac{2}{3} \\ 1 \end{pmatrix}, \qquad (21)$$

where,

$$A_1=\begin{pmatrix} 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 \\ \frac{5}{32} & \frac{27}{32} & 0 \end{pmatrix},\ A_2=\begin{pmatrix} 0 & 0 & 0 \\ \frac{2}{9} & 0 & 0 \\ \frac{-1}{160} & \frac{-9}{160} & 0 \end{pmatrix},\ A_3=\begin{pmatrix} 0 & 0 & 0 \\ \frac{4}{81} & 0 & 0 \\ \frac{-1}{480} & \frac{3}{160} & 0 \end{pmatrix},$$

$$B_1=\begin{pmatrix} \frac{5}{32} & \frac{27}{32} & 0 \end{pmatrix},\ B_2=\begin{pmatrix} \frac{-1}{160} & \frac{-9}{160} & 0 \end{pmatrix},\ B_3=\begin{pmatrix} \frac{-1}{480} & \frac{3}{160} & 0 \end{pmatrix}, U=\begin{pmatrix} 1 & 1 & 1 \end{pmatrix},\ V=1,\ Y=(Y_1,Y_2,Y_3)^T,$$

$F=(F_1,F_2,F_3)^T$, $F'=(F_1',F_2',F_3')^T$ and $F''=(F_1'',F_2'',F_3'')^T$.

The stability polynomial of this method (21) is

$$\Pi(w,z)=w-1-z-\frac{z^2}{2}-\frac{z^3}{6}-\frac{z^4}{24}-\frac{z^5}{720}-\frac{z^6}{1080}.$$

The interval of absolute stability of (21) is (-2.678, 0), see Fig. 2 for the plot. The (21) is nearly ARK-stable.

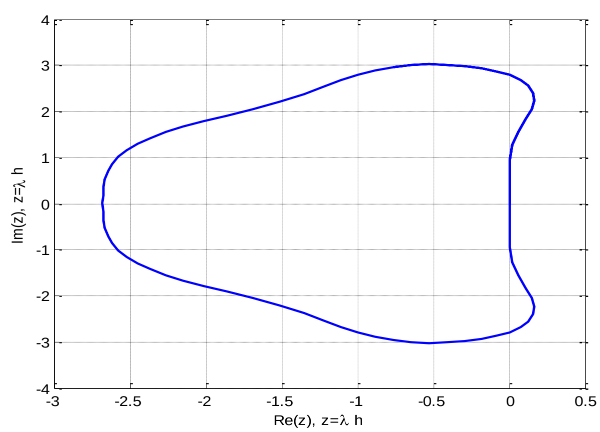

Fig. 2: The stability region of the GLM (21).

4.0 Numerical experiments and conclusion.

In this section we shall compare the numerical results obtained via error by using the TDGLM (17)and the third order RK method [5]

$$\begin{cases} Y_1 = y_{n-1} \ , \\ Y_2 = y_{n-1}.+\frac{2h}{3}F_1, \ F_1 = F(x_{n-1}, Y_1), \\ Y_3 = y_{n-1}.+\frac{h}{3}F_1+\frac{h}{3}F_2, \ F_2 = F\left(x_{n-1}+\frac{2}{3}h, Y_2\right), \\ y_{n+1} = y_{n-1}+\frac{h}{4}F_1+\frac{3h}{4}F_3, \ F_3 = F\left(x_{n-1}+\frac{2}{3}h, Y_3\right), \end{cases}$$

to solve the following IVPs:

Problem 1:

$$\begin{cases} y' = -10(y-1)^2, \ x \in [0, 20], \\ y(0) = 2, \ y(x) = 1/(1+10x). \end{cases}$$

Problem 2:

$$\begin{cases} y' = -2xy^2, \ x \in [0, 20], \\ y(0) = 1, \ y(x) = 1/(1+x^2). \end{cases}$$

Problem 3:

$$\begin{cases} y' = -y^{3/2}, \ x \in [0, 20], \\ y(0) = 1, \ y(x) = 1/(x+1)^{\frac{1}{2}}. \end{cases}$$

The method is implemented using constant step-size. The error: $|y(x_n) - y_n|$ in the solution when the methods are applied to problems 1-3 are given in Tables 1-3.

Table 1: ResultsforProblem 1 for comparison

h = 0.01			*h = 0.001*	
x	*TDGLMerror*	*RK error*	*TDGLM error*	*RK error*
5.0	5.56e - 07	5.39e - 05	1.06e - 08	5.07e - 06
10.0	1.42e - 07	1.59e - 05	2.71e - 09	1.51e - 06
15.0	6.36e - 08	7.73e - 06	1.21e - 09	7.37e - 07
20.0	3.59e - 08	4.59e -06	6.86e -10	4.39e -07

Table 2: ResultsforProblem 2 for comparison

h = 0.01			*h = 0.001*	
x	*TDGLM error*	*RK error*	*TDGLM error*	*RK error*
5.0	6.44e - 05	1.12e - 04	1.06e - 08	2.31e - 07
10.0	8.54e - 06	2.81e - 05	2.71e - 09	1.35e - 07
15.0	2.56e - 06	9.42e - 06	1.21e - 09	5.89e - 08
20.0	1.08e - 06	4.15e -06	6.86e -10	2.89e -08

Table 3:Results forProblem 3 for comparison

h = 0.01			*h = 0.001*	
x	*TDGLM error*	*RK error*	*TDGLM error*	*RK error*
5.0	1.57e - 11	2.40e - 09	1.66e - 16	2.38e - 12
10.0	6.36e - 12	9.87e - 10	9.99e - 16	9.79e - 13
15.0	3.62e - 12	5.64e - 10	7.77e - 16	5.59e - 13
20.0	2.41e - 12	3.76e -10	1.19e -16	3.72e -13

The numerical results in Tables 1-3 showthat the TDGLM (17) compared favourably with the classical third order RK methods in terms of accuracyon problems 1-3.

Thisimpressive performance is as a result of the RK stability property the TDGLM possesses.

In this paper we have summaries the construction of the TDGLM. Their stability polynomials are same as that of the classical RK methods in [5].Indeed, the intervals of absolute stabilities of these TDGLM are equivalent to that of the third order RK methods.

Reference:

[1] **J. C. Butcher,** On Runge Kutta processes of high order, J. Austral. Math. Soc., vol. IV, Part 2, (1964), 179-194.

[2] **J. C. Butcher,** General linear method: A SURVEY, Applied Numerical Mathematics}, 1, (1985), 273-284.

[3] **K. Burrage and J. C. Butcher,** Non-linear stability of a general class of differential equation methods.BIT, 20, (1980), 185-203.

[4] **J. C. Butcher,** The Numerical Analysis of Ordinary Differential Equation: Runge Kutta and General Linear Methods, Wiley, Chichester, 1987.

[5] **J. C. Butcher,** Numerical Methods for Ordinary Differential Equations. Second Edition. J. Wiley, Chichester, (2008).

[6] **E. Hairer, and G. Wanner,** Solving Ordinary Differential Equations II. Stiff and Differential-Algebraic Problems, Springer-Verlag, Berlin, (1996).

[7] **J. Hyo Jin Lee,** Numerical methods for Ordinary Differential Equations: A survey of some standard methods. M.Sc. Thesis. The University of Auckland (2004).

[8] **J. C. Butcher,** On the convergence of the numerical solutions to ordinary differential equations. Maths. Comp., 20, (1966), 1-10.

[9] **Ali Abdi and Gholamreza Hojjati,** Numerical solution of stiff ODEs using second derivative general linear methods. SciCADE, Toronto (2011), www.fields.utoronto.ca/programs/scientific/11-12/.../hojjati-talk.pdf.

[10]**J.C. Butcher and G. Hojjati,** Second derivative methods with RK stability, Numerical Algorithms, 40, (2005), 415–429.

[11] **J. C. Butcher and A. E. O'Sullivan,** Nordsieck methods with an off-step point. Numerical Algorithms, 31 (2002), 87-101.

[12] **R.I. Okuonghae,** Variable order explicit second derivative general linear methods. Comp. Applied Maths. Sociedade Brasileira de Matematica Aplicada e Computacional (SBMAC), (2013). See link.springer.com.

[13] **Okuonghae, R. I.** and Ikhile, M.N.O**,** Second Derivative General Linear Methods. Numerical Algorithms, December (2013). Online firstSee link.springer.com.See link.springer.com.

[14]**Ali Abdi and Gholamreza Hojjati,** An extension of general linear methods. Numerical Algorithms, 57, Issue 2, (2011), pp 149-167.

[15] **W.H. Enright,** Second derivative multistep methods for stiff ODEs. SIAM. J. Numer. Anal.} (1974), vol. 11 pp.321-331.

[16] **Okuonghae, R. I.** and Ikhile, M.N.O**,** Second derivative GLM with nearly ARK stability. J. Numerical Maths. (2013). Accepted for publication.

[17] **G. Dahlquist,** A special stability problem for linear multistep methods, BIT}, 3, (1963), pp 27-43.

[18] **W.M. Wright,** Explicit general linear methods with inherent Runge-Kutta stability. Numerical Algorithms. Vol 31, (2002), pp. 381-399.

[19] **R.I. Okuonghae and M.N.O. Ikhile,** A continuous formulation of A(α)-stable second derivative linear multistep methods for stiff IVPs and ODEs. J. of Algorithms and Comp. Technology, Vol. 6, No. 1 (2011), 79-101.