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On Rational Maps Whose Julia Set Is The Entire Complex Sphere.

K. B. Yuguda

Department of Mathematics and Computer Science,
NDA, PMB 2109, Kaduna.

Abstract

In this paper, following closely the technique of Ritt, we showed that for a class of rational maps, the Julia Set is the entire complex sphere. This is achieved by the use of a group of translations with two generators which makes it fairly easy to analyze the behavior of their iterates and of course, coupled with powerful techniques in complex analysis. Intuitively, if the Julia Set of a given set in this context is the entire complex sphere, then its Fatou Set is empty.

Keywords: equicontinuity, Fatou set, Julia set, lattice, period parallelogram, elliptic function.

1.0 Introduction

We recall that the dynamics of a rational map $\mathcal{R}$ induces a subdivision of the complex sphere into two sets, namely, the Fatou Set F and the Julia Set J of $\mathcal{R}$ and then introdu ce the notion of equicontinuity for a family of maps. We begin with the definition of continuity which serve to motivate the crucial notion of equicontinuity.

Definition 1.1 A map $f : (\mathrm{X}, d) \to (X_1, d_1)$ is continuous at x_0 *in* X if for every $\varepsilon > 0$, $\exists$ $\delta > 0$, $\ni$ for every x, $d(x_0 , x) < \delta$, whenever $d_1(f(x_0), f(x)) < \varepsilon$.

Indeed δ depends on f, x_0 and ε, but if δ can be found so that this holds for all x and for all f in some family $\mathcal{F}$ of maps of X into Y, then we say that the family $\mathcal{F}$ is equicontinuous at x_0. This is the formal expression for the idea of preservation of proximity.

Definition 1.2 A family $\mathcal{F}$ of maps of (X,d) into (X_1, d_1) is equicontinuous at x_0 *if and only if* for every $\varepsilon > 0$, $\exists$ $\delta > 0, \ni \forall\, x \in X$ *and* $f \epsilon \mathcal{F}$,

$$d(x_0, x) < \delta \text{ , whenever } d_1\big(f(x_0), f(x)\big) < \varepsilon.$$

Intuitively the family $\mathcal{F}$ is equicontinuous on a set X if it is equicontinuous at each point x_0 of X.

By contrast, if the family $\mathcal{F}$ is equicontinuous on each of the subsets D_α of X, then it is automatically equicontinuous on the union $\cup\, D_\alpha$. Taking the collection $\{D_\alpha\}$ to be the class of all open subsets of X on which $\mathcal{F}$ is equicontinuous, we are then lead to the following general principle ([1] & [2]), which provides us with the formal definition of the Fatou and Julia sets of a rational map $\mathcal{R}$ named after the founding fathers Fatou and Julia.

Definition 1.3 Let $\mathcal{R}$ be a non constant rational map. The Fatou set of $\mathcal{R}$ is the maximal open subset of $\mathbb{C}_\infty$ on which $\{\mathcal{R}^n\}$ is equicontinuous and its complement in $\mathbb{C}_\infty$ the Julia set of $\mathcal{R}$.

Although the use of the term "Julia Set" is standard, the use of "Fatou Set" was suggested as late as 1984, [3]. It seems appropriate, but the reader should be familiar with the common alternatives, namely the stable set and the set of normality. This latter term is a reference to normal families of analytic functions discussed elsewhere, ([2] , [4] & [5]). However, because of its general appeal we have preferred to base our definition on equicontinuity.

Corresponding author: E-mail: kyuguda@gmail.com, Tel.: +2348033691107

Denoting the Fatou and Julia Sets of $\mathcal{R}$ by $F(\mathcal{R})$ and J($\mathcal{R}$) respectively, we note that, by definition, F($\mathcal{R}$) is open, and J ($\mathcal{R}$) is compact and that both sets are completely invariant under $\mathcal{R}$.We refer the interested reader to the details, ([6],[7],[8] & [9]).

2.0 Statement and proof of main result

We begin this section by quoting a result of Latte's [10] which is central to our work and which asserts that the Julia set of the rational map

$$f : z \to \frac{(z^2+1)^2}{4z(z^2-1)}, \tag{1.1}$$

is the entire complex sphere. Intuitively, by **Definition 1.3** the compliment of the entire complex sphere, in this case the Fatou Set, is empty.

We then make more general observations about other rational functions with this property and go on to prove the assertion in a slightly different but broader perspective.

In this regard we shall exploit the unique technique of a group of translations with two generators despite its demand for rigorous complex analysis.

Let λ *and* μ be complex numbers that are not real multiples of each other, and let Λ be the corresponding lattice;

$$\Lambda = \{m\lambda + n\mu : n, m\epsilon\mathbb{Z}\} \tag{1.2}$$

A period parallelogram for Λ is any closed parallelogram of the form;

$$\Omega = \{\mathrm{z} + \mathrm{s}\lambda + \mathrm{t}\mu : 0 \leq \mathrm{s} \leq 1, 0 \leq \mathrm{t} \leq 1\} \tag{1.3}$$

We then note that a non constant function f *is* an elliptic function for Λ if it is meromorphic on $\mathbb{C}$, and if each ω in Λ is a period of f; that is if for all z in $\mathbb{C}$ and all ω inΛ, $f(z+\omega) = f(z)$. Of course any such function maps $\mathbb{C}$ *into* $\mathbb{C}_\infty$ and since $f(\mathbb{C}) = f(\Omega)$, we see that $f(\mathbb{C})$ is compact subset of $\mathbb{C}_\infty$. However, by the open mapping theorem, $f(\mathbb{C})$ is an open set of $\mathbb{C}_\infty$. Thus, $f(\Omega) = f(\mathbb{C}) = \mathbb{C}_\infty$.

Our argument is based on the relevance of the Weierstrass elliptic function and its rich properties given in (1.4)

$$\wp(z) = \frac{1}{z^2} + \sum{}' \left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] \tag{1.4}$$

where $\sum'$ denotes summation over non-zero elements ω in Λ, ([4], pp 272-276, and [11 - 13].)

It is clear that $\wp$(z) and its derivatives satisfy certain algebraic identities inclusive of

$$\wp(2z) = \mathcal{R}(\wp(z)) \tag{1.5}$$

where $\mathcal{R}$ is the rational function given by;

$$\mathcal{R}(z) = \frac{z^4 + g_2\frac{z^2}{2} + 2g_3 z + (\frac{g_2}{4})^2}{4z^3 - g_2 z - g_3} \tag{1.6}$$

And g_2 *and* g_3 are known quantities defined in terms of the lattice Λ..

We now go on to prove the result in a slightly different but broader perspective .In this regard we shall require a lemma of Lang [12].

Lemma 2.1 Let D be any disc in $\mathbb{C}$, let $\mathsf{U} = \wp^{-1}(D)$, define $\Phi(z) = 2z$. As U is open, and as $\Phi^n(\mathsf{U})$ is the set U expanded by a factor 2^n, then for sufficiently large n, $\Phi^n(\mathsf{U})$ contains a period parallelogram Ω of $\wp$.

Proof. We can see from (1.5) that for these n,

$$\mathcal{R}^n(D) = \mathcal{R}^n(\wp(\mathsf{U})) = \wp(2^n\mathsf{U}) = \mathbb{C}_\infty. \tag{1.7}$$

And by a result of Blanchard [3], this implies that the family $\mathcal{R}^n$ explode any small disc D onto the whole sphere. But as D is arbitrary, (1.7) implies that the family $\{\mathcal{R}^n\}$ is not equicontinuous on any open subset of the complex sphere and so we deduce that $J(\mathcal{R}) = \mathbb{C}_\infty$. By this argument, we therefore have a family of rational maps whose Julia set is the sphere, consequently we have;

$$(g_2)^3 - 27(g_3)^2 \neq 0 \qquad (1.8).$$

In order to justify (1.2), we shall construct Λ so that $g_2 = 4$ *and* $g_3 = 0$, by using the lattice Λ in (1.4) as the ideal point of departure.

Let

$$S_n(\Lambda) = \Sigma' \omega^{-n}, \qquad (1.9)$$

This series clearly converges for $n \geq 3$, and as the general term for $\wp$ is $O(\omega^{-3})$,we see that $\wp$ is meromorphic in $\mathbb{C}$. Intuitively the quotient space $\mathbb{C}/\Lambda$ (which is also a quotient group) is topologically a torus, and as every element of Λ is a period of $\wp$, we see that $\wp$ induces a map $\wp_o$ of $\mathbb{C}/\Lambda$ onto $\mathbb{C}_\infty$.

As these are compact Riemann Surfaces, $\wp_0$ is an N-fold covering map for some N, and by considering the poles of $\wp$, we see that N=2. It then follows that for each ω in $\mathbb{C}_\infty$, there are exactly two solutions (modulo Λ) of the equation $\wp(z) = \omega$ in $\mathbb{C}$. Without loss of generality these can be taken to be say , u and $(\lambda + \mu) - u$. As $\wp(2[\lambda + \mu - u]) = \wp(2u)$, we can define a map $\omega \to \wp(2u)$ of $\mathbb{C}_\infty$ onto itself which is independent of the choice of u. It is easy to see that this map is analytic and must be a rational map $\mathcal{R}$, it then follows that $\mathcal{R}(\wp(u)) = \wp(2u)$.

Without loss of generality we can replace 2 here by any other integer and indeed by certain other numbers as well, ([7] , [12] & [14]).

3.0 Now the proof of main result.

We observe here that the derivative $\wp'$ has triple poles at, and only at, the points in Λ, and because of this, it is not hard to construct a cubic polynomial P such that the elliptic function $\wp'(z)^2$ - p($\wp(z)$) has no poles at the origin, and hence no poles in $\mathbb{C}$. Such an elliptic function must be constant (by Liouville's Theorem, for it is bounded on any period parallelogram, and hence also on $\mathbb{C}$), and a computation of P leads to the relation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \qquad (1.10)$$

where the g_j are given by

$$g_2 = 60S_4(\Lambda) \quad \text{and} \quad g_3 = 140S_6(\Lambda) \qquad (1.11)$$

By selecting distinct points u and v in $\mathbb{C}$ at which $\wp$ has different values and determining A and B so that $\wp'(u) = A\wp(u) + B$ and $\wp'(v) = A\wp(v) + B$, then it becomes apparent that the elliptic function $f(z) = \wp'(z) - A\wp(z) - B$ has three poles at (and only at) each point in Λ, and as a consequence of this , f must have three zeros. Intuitively, two of these zeros are at u and v respectively. Generally however, if an elliptic function has poles p_i *and zeros* z_j in a period parallelogram, then $\sum p_i$ differs from $\sum z_j$ by an element of Λ. In this case, all the poles of f occur at the origin, thus the zeros of f must sum up to zero (modulo Λ) and so must be u , v and $-(u+v)$ and their translates by Λ. However, as

$[f(z) + A\wp(z) +]^2 = \wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, we find that $\wp(u), \wp(v)$ *and* $\wp(-[u+v])$ are distinct solutions of the equation

$$\wp(u) + \wp(v) + \wp(-[u+v]) = \frac{A^2}{4} = \frac{1}{4}\left(\frac{\wp'(u) - \wp(v)}{\wp(u) - \wp(v)}\right)^2.$$

Letting $u \to v$, and using the fact that $\wp$ is an even function, we now obtain

$$2\wp(v) + \wp(2v) = \frac{1}{4}\,[\wp''(v)/\wp'(v)]^2 \qquad (1.12)$$

Finally, differentiating both sides of (1.10) gives an expression for $\wp''(z)$, and using same in (1.10) and (1.12), we obtain the addition formula given in (1.5) and (1.6). Now with $\lambda = \tau$ and $\mu = i\tau$ where $\tau > 0$, we have $i\Lambda = \Lambda$ and so from (1.9), $S_n(\Lambda) = S_n(i\Lambda) = i^{-n}\, S_n(\Lambda)$.

Putting $n = 6$, we find from (1.11) that $g_3 = 0$. It remains to show that with the general lattice Λ, the derivative $\wp'$ is zero at each of the points $\lambda/2$, $\mu/2$ and $(\lambda + \mu)/2$.

Indeed for each z, $\wp(\lambda - z) = \wp(z)$ and equating the derivative of each side at $\lambda/2$, we find that $\wp'\left(\frac{\lambda}{2}\right) = 0$. The same argument holds for the points $\mu/2$ and $(\lambda + \mu)/2$.

If we put $e_1 = \wp\left(\frac{\lambda}{2}\right)$, $e_2 = \wp\left(\frac{\mu}{2}\right)$, *and* $e_3 = \wp\left(\frac{[\lambda+\mu]}{2}\right)$, then from (1.9) coupled with the fact that any family of maps whose Julia set is the sphere, we always have $g_2{}^3$ -$27g_3{}^2 \neq 0$. Consequently any pair (g_2, g_3) satisfying this relation is realized by some lattice, ([12] ,pp37). Thus we see that e_1, e_2 *and* e_3 are distinct zeros of the cubic equation $4z^3$- g_2z - g_3 = 0. In particular $e_1 + e_2 + e_3 = 0$, $e_2e_3 + e_3e_1 + e_1e_2 = -g_{2/4}$, $e_1e_2e_3 = g_{3/4}$ which simplifies considerably when $g_3 = 0$. Indeed in this case, some e_j is zero and (1.6) reduces to $(z^2 + \alpha^2)^2/4z(z^2 - \alpha^2)$. This then completes the details of the proof.

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