Journal of the Nigerian Association of Mathematical Physics
***Volume 26*, (*March, 2014*), pp 28 – 35**
© J. of NAMP

Derivation of an Optimal Expression for Solution Matrices of a Class of Single – Delay Scalar Differential Equations.

C. Ukwu And E.J.D. Garba

Department of Mathematics
University of Jos, P.M.B 2084, Jos, Nigeria.

Abstract

This paper derived optimal expressions for solution matrices for single – delay autonomous linear differential equations with an accompanying corollary, on arbitrary intervals of length equal to the delay h , for non –negative time periods. The formulation and the development of the main result exploited an earlier work in [1] on the interval [0, 4h]. The proof was achieved using combinations of summation notations, integrals, change of variables technique, as well as the method of steps to obtain these matrices on successive intervals of length equal to the delay h. By exploiting above results, the paper obtained the solution of an initial function problem, as well as interrogated its smoothness disposition. The obtained results globally extend the time scope of applications of solution matrices to the solutions of initial function problems, rank conditions for controllability and cores of targets, constructions of controllability Grammians and admissible controls for transfers of points associated with controllability problems.

1.0 Introduction

The qualitative approach to the controllability of functional differential control systems have been areas of active research for the past fifty years among control theorists and applied mathematicians in general. This circumvents the severe difficulties associated with the search for and computations of solutions of such systems. Unfortunately computations of solutions cannot be wished away in the tracking of trajectories and many practical applications. Literature on state space approach to control studies is replete with variation of constants formulas, which incorporate the solution matrices of the free part of the systems [2 - 9]. Regrettably no author has made any attempt to obtain general expressions for such solution matrices or special cases of such matrices involving the delay, h. The usual approach thus far is to start from the interval $[0,h]$ and compute the solution matrices and solutions for given problem instances and then use the method of steps to extend these to the intervals $[kh,(k+1)h]$, for positive integral k, not exceeding 2, for the most part [10] and [8]. Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary k. In other words such approach fails to address the issue of the structure of solution matrices and solutions quite vital for real-world applications. The need to address such short-comings has become imperative; this is the major contribution of this paper, with limitations to scalar equations and wide-ranging implications for extensions to systems and holistic approach to controllability studies.

2.0 Theoretical Analysis

We consider the class of single-delay differential equations:

$$\dot{x}(t)=ax(t)+bx(t-h),\ t\in\mathbf{R} \tag{1}$$

where a and b are arbitrary constants. Let $Y_{k-i}(t-ih)$ be a solution matrix of (1) on the interval $J_{k-i}=[(k-i)h,(k+1-i)h], k\in\{0,1,\cdots\}, i\in\{0,1\}$, where

Corresponding author: ***C. Ukwu,*** E-mail: cukwu@hotmail.com, Tel.: +2347052765751

$$Y(t)=\begin{cases}1, \ t=0,\\ 0, \ t<0.\end{cases}$$

Note that $Y(t)$ is a generic solution matrix for any $t\in\mathbf{R}$. The solution matrices will be obtained piece – wise on successive intervals of length h.

Preliminary lemma:

$$Y(t)=\begin{cases}e^{at}, \ t\in J_0; & (2)\\ e^{at}+b(t-h)e^{a(t-h)} \ \ t\in J_1; & (3)\\ e^{at}+b(t-h)e^{a(t-h)}+\dfrac{b(t-2h)^2e^{a(t-2h)}}{2!}, t\in J_2; & (4)\end{cases}$$

Proof

On $(0,h),\ Y(t-h)=0\Rightarrow \dot{Y}(t)=aY(t)$ a.e. on $[0,h]\Rightarrow Y(t)\equiv Y_1(t)=e^{at}c; Y(0)=1\Rightarrow c=1$

$$\Rightarrow Y(t)=e^{At} \text{ on } J_0=[0,h]. \qquad (5)$$

Consider the interval $[h,2h]$. Then on $(h,2h)$,

$$t-h\in(0,h)\Rightarrow \dot{Y}(t)=aY(t)+be^{a(t-h)}\Rightarrow \frac{d}{dt}\left[e^{-at}Y(t)\right]=e^{-at}\left[\dot{Y}(t)-aY(t)\right]=be^{a(t-h)}$$

$$\Rightarrow \frac{d}{dt}\left[e^{-at}Y(t)\right]=e^{-at}\left[\dot{Y}(t)-aY(t)\right]=e^{-at}be^{a(t-h)}\Rightarrow e^{-at}Y(t)-e^{-ah}Y(h)=\int_h^t e^{-as}be^{a(s-h)}ds$$

$$\Rightarrow Y(t)=e^{a(t-h)}Y(h)+\int_h^t e^{a(t-s)}be^{a(s-h)}ds, \text{ on } [h,2h].$$

$$Y(h)=e^{ah}\Rightarrow Y(t)=e^{at}+\int_h^t e^{a(t-s)}be^{a(s-h)}ds=e^{at}+b(t-h)e^{a(t-h)},\ t\in J_1; \qquad (6)$$

Consider the interval $J_2=[2h,3h]$; then $t\in[2h,3h]\Rightarrow t-h\in[h,2h]\Rightarrow s_2-h\in[h,2h]$.

By the continuity of $Y(t)$, $Y(2h)=e^{2ha}+\int_h^{2h} e^{a(2h-s_1)}be^{a(s_1-h)}ds_1$. Therefore,

$$t\in(2h,\ 3h)\Rightarrow \dot{Y}(t)-aY(t)=bY(t-h)\Rightarrow e^{at}\frac{d}{dt}\left[e^{-at}Y(t)\right]=bY(t-h)$$

$$\Rightarrow Y(t)=e^{a(t-2h)}Y(2h)+\int_{2h}^t e^{a(t-s_2)}bY(s_2-h)ds_2$$

$$\Rightarrow Y(t)=e^{a(t-2h)}Y(2h)+\int_{2h}^t e^{a(t-s_2)}\left[be^{A_0(s_2-h)}+\int_h^{s_2-h} e^{a(s_2-s_1-h)}be^{a(s_1-h)}ds_1\right]ds_2, \text{ on } J_2.$$

$$Y(t)\ =\ e^{at}+\int_h^{2h} e^{a(t-s_1)}be^{a(s_1-h)}ds_1+\int_{2h}^t e^{a(t-s_2)}be^{a(s_2-h)}ds_2+\int_{2h}^t\int_h^{s_2-h} e^{a(t-s_2)}be^{a(s_2-s_1-h)}be^{a(s_1-h)}ds_1ds_2$$

$$\Rightarrow Y(t)=e^{at}+\int_h^t e^{a(t-s)}be^{a(s-h)}ds+\int_{2h}^t\int_h^{s_2-h} e^{a(t-s_2)}be^{a(s_2-s_1-h)}be^{a(s_1-h)}ds_1ds_2$$

$$= e^{at} + b(t-h)e^{a(t-h)} + \frac{b(t-2h)^2 e^{a(t-2h)}}{2!}, t \in J_2. \qquad (7)$$

This completes the proof.
From our investigation of emerging patterns for $Y(t)$ on $[0,\ 3h]$, we state as follows:

Theorem

$$Y(t) = \begin{cases} e^{at},\ t \in J_0 & (8) \\ e^{at} + \sum_{i=1}^{k} e^{a(t-ih)} \frac{b^i}{i!}(t-ih)^i, t \in J_k,\ k \geq 1. & (9) \end{cases}$$

Proof
The theorem is valid for $k \in \{0,1\}$ as earlier established. Assume the validity of the theorem for $0 \leq k \leq p$ for some integer $p \geq 2$. Consider the interval $J_{p+1}, p \geq 1$. Then $t-h \in J_p$. Hence

$$Y(t-h) = e^{a(t-h)} + \sum_{i=1}^{p} e^{a(t-h-ih)} \frac{b^i}{i!}(t-h-ih)^i,\ t-h \in J_p, p \geq 1, \qquad (10)$$

by the induction hypothesis. Hence on

$$J_{p+1}, e^{at}\frac{d}{dt}\left[e^{-at}Y(t)\right] = \dot{Y}(t) - aY(t) = bY(t-h) \Rightarrow \frac{d}{dt}\left[e^{-at}Y(t)\right] = be^{-at}Y(t-h) \Rightarrow$$

$$\left[e^{-at}Y(s)\right]_{s=(p+1)h}^{t} = \int_{(p+1)h}^{t} be^{-as}Y(s-h)ds \Rightarrow Y(t) = e^{a(t-[p+1]h)}Y([p+1]h) + \int_{(p+1)h}^{t} be^{a(t-s)}Y(s-h)ds$$

$$\Rightarrow Y(t) = e^{a(t-[p+1]h)}\left[e^{a(p+1)h} + \sum_{i=1}^{p} e^{a([p+1-i]h}\frac{b^i}{i!}\left([p+1-i]h\right)^i\right]$$

$$+ \int_{(p+1)h}^{t} be^{a(t-s)}\left[e^{a(s-h)} + \sum_{i=1}^{p} e^{a(s-[i+1]h)}\frac{b^i}{i!}\left(s-[i+1]h\right)^i\right]ds \qquad (11)$$

$$\Rightarrow Y(t) = \left[e^{at} + \sum_{i=1}^{p} e^{a(t-ih)}\frac{b^i}{i!}\left([p+1-i]h\right)^i\right]$$

$$+ \int_{(p+1)h}^{t}\left[be^{a(t-h)} + \sum_{i=1}^{p} e^{a(t-[i+1]h)}\frac{b^{i+1}}{i!}\left(s-[i+1]h\right)^i\right]ds \qquad (12)$$

$$\Rightarrow Y(t) = \left[e^{at} + \sum_{i=1}^{p} e^{a(t-ih)}\frac{b^i}{i!}\left([p+1-i]h\right)^i\right] + be^{a(t-h)}\left(t-[p+1]h\right)$$

$$+\left[\sum_{i=1}^{p} e^{a(t-[i+1]h)}\frac{b^{i+1}}{(i+1)!}\left[\left(t-[i+1]h\right)^{i+1} - \left([p+1]h-[i+1]h\right)^{i+1}\right]\right] \qquad (13)$$

$$\Rightarrow Y(t)=\left[e^{at}+\sum_{i=1}^{p}e^{a(t-ih)}\frac{b^i}{i!}\left([p+1-i]h\right)^i\right]+be^{a(t-h)}\left(t-[p+1]h\right) \tag{14}$$

$$+\left[\sum_{i=2}^{p+1}e^{a(t-ih)}\frac{b^i}{i!}\left[\left(t-ih\right)^i-\left([p+1-i]h\right)^i\right]\right] \text{(using change of variables technique)}$$

Therefore,

$$Y(t)=e^{at}+\sum_{i=1}^{p}e^{a(t-ih)}\frac{b^i}{i!}\left([p+1-i]h\right)^i+b\left(t-[p+1]h\right)e^{a(t-h)}+\sum_{i=2}^{p+1}e^{a(t-ih)}\frac{b^i}{i!}\left(t-ih\right)^i$$

$$-\sum_{i=2}^{p+1}\frac{b^i}{i!}\left([p+1-i]h\right)^i e^{a(t-ih)} \tag{15}$$

$$=e^{at}+\sum_{i=1}^{p+1}e^{a(t-ih)}\frac{b^i}{i!}\left([p+1-i]h\right)^i-0+b\left(t-[p+1]h\right)e^{a(t-h)}+\sum_{i=1}^{p+1}e^{a(t-ih)}\frac{b^i}{i!}\left(t-ih\right)^i-b\left(t-h\right)e^{a(t-h)}$$

$$-\sum_{i=1}^{p+1}e^{a(t-ih)}\frac{b^i}{i!}\left([p+1-i]h\right)^i+b\left([p+1-1]h\right)e^{a(t-h)}.\ \text{Clearly } -b(t-h)+bph=-b(t-[p+1]h).$$

Therefore, $Y(t)=e^{at}+\sum_{i=1}^{p+1}e^{a(t-ih)}\frac{b^i}{i!}(t-ih)^i, t\in I_{p+1}$, completing the proof of the theorem.

The usefulness of this theorem can be seen from its application in the variation of constants formula to obtain the solutions of initial function problems of scalar type. Consider the following problem:

$$\dot{x}\left(t\right)=ax\left(t\right)+bx\left(t-1\right), t\in[0,\ 2] \tag{16}$$

$$x\left(t\right)=\phi\left(t\right)=1+t, t\in\left[-1,\ 0\right] \tag{17}$$

The Variation of Constants formula for (1) is given by:

$$x(t)=Y(t)\phi(0)+\int_{-h}^{0}Y(t-s-h)A_1\phi(s)ds, t\geq 0 \tag{18}$$

There is no direct straight-forward application of the above formula in one fell swoop– a fact that is hardly emphasized by control practitioners. The method of steps must be applied by reasoning as follows:

$h=1;\ s\in[-1,0]\Rightarrow -s-1\in[-1,\ 0];\ t\in J_0\Rightarrow t-s-1\in[-1,\ 1];\ 0\leq t-s-1\leq 1 \text{ iff } t-2\leq s\leq t-1$

$\Rightarrow -1\leq s\leq t-1,$ for the feasibility of s.

$$x(t)=Y(t)\phi(0)+b\int_{-1}^{t-1}Y(t-s-h)\phi(s)ds, t\in J_0 \tag{19}$$

$$\Rightarrow x(t)=e^{at}+b\int_{-1}^{t-1}e^{a(t-s-1)}(1+s)ds, t\in J_0 \tag{20}$$

$$\Rightarrow x(t)=e^{at}+be^{a(t-1)}\left[\int_{-1}^{t-1}e^{-as}ds+\int_{-1}^{t-1}e^{-as}sds\right]$$

$$=e^{at}+\frac{b}{a}e^{a(t-1)}\left[e^{-as}\right]_{-1}^{t-1}+be^{a(t-1)}\left[-\frac{1}{a}se^{-as}-\frac{1}{a^2}e^{-as}\right]_{-1}^{t-1}$$

$$= e^{at} - \frac{b}{a}e^{a(t-1)}\left[e^{-a(t-1)} - e^{a}\right] - be^{a(t-1)}\left[-\frac{1}{a}(t-1)e^{-a(t-1)} + \frac{e^{a}}{a}\right] - be^{a(t-1)}\left[-\frac{1}{a^2}e^{-a(t-1)} - \frac{e^{a}}{a}\right]$$

$$\Rightarrow x(t) = e^{at} - \frac{b}{a^2} - \frac{b}{a}t - \frac{b}{a^2} + \frac{b}{a^2}e^{at} = \frac{b}{a^2} - \frac{b}{a}t + \left(1 + \frac{b}{a^2}\right)e^{at}, \text{on } J_0 \qquad (21)$$

The following analysis is imperative for $h = 1,\ s \in [-1,\ 0]$ and $t \in J_1$:

$h = 1;\ s \in [-1,0] \Rightarrow -s-1 \in [0,1];\ t \in J_1 \Rightarrow t-s-1 \in [0,2];\ t-s-1 \geq 1$ iff $s \leq t-2$. Thus

$t-s-1 \geq 1$ iff $s \in [-1, t-2]$, and $t-s-1 \leq 1$ iff $s \in [t-2, 0]$.

$0 \leq t-s-1 \leq 1 \Rightarrow Y(t-s-1) = e^{a(t-s-1)}$;

$1 \leq t-s-1 \leq 2 \Rightarrow Y(t-s-1) = e^{a(t-s-1)} + b(t-s-2)e^{a(t-s-2)}$. We deduce from $\phi(0) = 1$ that

$$x(t) = e^{at} + b(t-1)e^{a(t-1)} + \int_{t-2}^{0} be^{a(t-s-1)}(1+s)ds + \int_{-1}^{t-2} b\left[e^{a(t-s-1)} + b(t-s-2)e^{a(t-s-2)}\right](1+s)ds$$

$$= e^{at} + b(t-1)e^{a(t-1)} + be^{a(t-1)}\int_{t-2}^{0} e^{-as}ds + be^{a(t-1)}\int_{t-2}^{0} se^{-as}ds + be^{a(t-1)}\int_{-1}^{t-2} e^{-as}ds + b^2te^{a(t-2)}\int_{-1}^{t-2} e^{-as}sds$$

$$-b^2e^{a(t-2)}\int_{-1}^{t-2} se^{-as}ds - 2b^2e^{a(t-2)}\int_{t-2}^{0} e^{-as}ds + be^{a(t-1)}\int_{-1}^{t-2} se^{-as}ds + b^2te^{a(t-2)}\int_{-1}^{t-2} se^{-as}ds$$

$$-b^2e^{a(t-2)}\int_{-1}^{t-2} s^2e^{-as}ds - 2b^2e^{a(t-2)}\int_{t-2}^{0} se^{-as}ds = e^{at} + b(t-1)e^{a(t-1)} + be^{a(t-1)}\int_{-1}^{0} e^{-as}ds + be^{a(t-1)}\int_{-1}^{0} se^{-as}ds$$

$$+b^2te^{a(t-2)}\int_{-1}^{t-2} e^{-as}ds - 3b^2e^{a(t-2)}\int_{-1}^{t-2} se^{-as}ds - 2b^2e^{a(t-2)}\int_{-1}^{t-2} e^{-as}ds - b^2te^{a(t-2)}\int_{-1}^{t-2} se^{-as}ds$$

$$-b^2e^{a(t-2)}\int_{-1}^{t-2} s^2e^{-as}ds = e^{at} + b(t-1)e^{a(t-1)} - \frac{b}{a}e^{a(t-1)}\left[1 + e^{a}\right] - be^{a(t-1)}\left[\frac{1}{a^2} + \frac{1}{a}e^{a} - \frac{1}{a^2}e^{a}\right]$$

$$-\frac{b}{a}te^{a(t-2)}\left[e^{a(t-2)} - e^{a}\right] + 3b^2e^{a(t-2)}\left[\frac{1}{a}(t-2)e^{-a(t-2)} + \frac{1}{a}e^{a} + \frac{1}{a^2}e^{-a(t-2)} - \frac{1}{a^2}e^{a}\right]$$

$$+2\frac{b^2}{a}e^{a(t-2)}\left[e^{a(t-2)} - e^{a}\right] - b^2te^{a(t-2)}\left[\frac{1}{a}(t-2)e^{-a(t-2)} + \frac{1}{a}e^{a} + \frac{1}{a^2}e^{-a(t-2)} - \frac{1}{a^2}e^{a}\right]$$

$$+ \frac{b^2}{a}e^{a(t-2)}\left[s^2e^{-as}\right]_{-1}^{t-2} - 2\frac{b^2}{a}e^{a(t-2)}\int_{-1}^{t-2} se^{-as}ds.$$

The last integral $\frac{b^2}{a}e^{a(t-2)}\left[s^2e^{-as}\right]_{-1}^{t-2} - 2\frac{b^2}{a}e^{a(t-2)}\int_{-1}^{t-2} se^{-as}ds = 2\frac{b^2}{a}e^{a(t-2)}\left[\frac{1}{a}se^{-as} + \frac{1}{a^2}e^{-as}\right]_{-1}^{t-2}$

$$\Rightarrow x(t) = e^{at} + be^{a(t-1)} + bte^{a(t-1)} - \frac{b}{a}e^{a(t-1)} + \frac{b}{a}e^{at} - \frac{b}{a^2}e^{a(t-1)} - \frac{b}{a}e^{at} + \frac{b}{a^2}e^{at} - \frac{b^2}{a}t + \frac{b^2}{a}te^{a(t-1)}$$

$$+3b^2(t-2) + 3\frac{b^2}{a}e^{a(t-1)} + 3\frac{b^2}{a} - 3\frac{b^2}{a^2}e^{a(t-1)} + 2\frac{b^2}{a} + 2\frac{b^2}{a}e^{a(t-1)} - \frac{b^2}{a}t^2 + 2\frac{b^2}{a}t - \frac{b^2}{a}te^{a(t-1)}$$

$$-\frac{b}{a^2}t + \frac{b^2}{a^2}te^{a(t-1)} + \frac{b^2}{a}t^2 - 4\frac{b^2}{a}t + 4\frac{b^2}{a} - \frac{b^2}{a}e^{a(t-1)} + 2\frac{b^2}{a^2} - 4\frac{b^2}{a^2} + 2\frac{b^2}{a^2} + 2\frac{b^2}{a^2}e^{a(t-1)} - 2\frac{b^2}{a^3}e^{a(t-1)}$$

$$= 2\frac{b^2}{a^3} - \frac{b^2}{a^2} + \frac{b^2}{a^2}t + e^{at}\left(1+\frac{b}{a^2}\right) - \left(b + \frac{b}{a} + \frac{b}{a^2} + \frac{b^2}{a^2}\right)e^{a(t-1)} - 2\frac{b^2}{a^3}e^{a(t-1)} + \left(b + \frac{b^2}{a^2}\right)te^{a(t-1)}.$$

Interrogation of smoothness of solutions

Let $y_k(t)$ denote the solution of (16) and (17) on $J_{k-1}, k \in \{1,2,\cdots\}$; then $a \neq 0$

$\Rightarrow \lim_{t\to 0^+} \dot{y}_1(t) = -\frac{b}{a} + a\left(1+\frac{b}{a^2}\right) = a;\ \lim_{t\to 0^-}\dot{\phi}(t) = 1 \Rightarrow$ non-smoothness of the solution $t = 0$.

$\lim_{t\to 1^+} \dot{y}_2(t) = ab - \frac{b}{a} + a\left(1+\frac{b}{a^2}\right)e^a;\ \lim_{t\to 1^-}\dot{y}_1(t) = -\frac{b}{a} + a\left(1+\frac{b}{a^2}\right)e^a \Rightarrow \lim_{t\to 1^+}\dot{y}_2(t) = \lim_{t\to 1^-}\dot{y}_1(t) + ab \quad \Rightarrow$ the solution is not differentiable at $t = 1 \Rightarrow$ lack of smoothness property at $t = 1$.

Case $a = 0$.

$a = 0 \Rightarrow \dot{x}(t) = bx(t-1),\ t \ge 0;\ x(t) = \phi(t) = 1+t,\ t\in[-1,0]$. Hence

$\dot{x}(t) = bt \Rightarrow x(t) = \frac{1}{2}bt^2 + d_1, t\in J_0$. The continuity condition $x(0) = \phi(0) = 1$

$\Rightarrow d_1 = 1 \Rightarrow x(t) \equiv y_1(t) = \frac{1}{2}bt^2 + 1$ on J_0.

On (1, 2), $\dot{x}(t) = \frac{1}{2}b^2(t-1)^2 + b \Rightarrow x(t) \equiv y_2(t) = \frac{1}{6}b^2(t-1)^3 + bt + d_2$. The continuity condition

$y_2(1) = y_1(1) \Rightarrow d_2 = 1 - \frac{1}{2}b \Rightarrow y_2(t) = \frac{1}{6}b^2(t-1)^3 + bt - \frac{b}{2} + 1$, on J_1. $\lim_{t\to 1^+}\dot{y}_2(t) = b; \lim_{t\to 1^-}\dot{y}_1(t) = b \Rightarrow$ the solution has the derivative b at $t = 1$.

Also $\lim_{t\to 0^+}\dot{y}_1(t) = 0;\ \lim_{t\to 1^-}\dot{\phi}(t) = 1 \Rightarrow$ the solution has no derivative at $t = 0$.

We proceed to investigate the analyticity or otherwise of the solution matrices. We reason as follows: for arbitrary delay $h > 0$,

$Y(t) = e^{at}$, on $[0,\ h]; \dot{Y}(t) = ae^{at}$, on $(0,h); Y(t) = e^{at} + b(t-h)e^{a(t-h)}$ on $[h, 2h]$

$\Rightarrow \dot{Y}(t) = ae^{at} + b\left(1 - ah + at\right)e^{a(t-h)}; \lim_{t\to h^-}\dot{Y}(t) = ae^{ah}, \lim_{t\to h^+}\dot{Y}(t) = ae^{ah} + b \Rightarrow \lim_{t\to h^-}\dot{Y}(t) \neq \lim_{t\to h^+}\dot{Y}(t)$.

Therefore $\dot{Y}(t)$ does not exist at $t = h$. In other words $Y(t)$ is not differentiable at $t = h$ and hence not analytic there. Also $Y(t) = 0$, for $t < 0 \Rightarrow \dot{Y}(t) = 0$, on $(-\infty, 0) \Rightarrow \lim_{t\to 0^-}\dot{Y}(t) = 0;\ \lim_{t\to 0^+}\dot{Y}(t) = a \Rightarrow \lim_{t\to 0^-}\dot{Y}(t) \neq \lim_{t\to 0^+}\dot{Y}(t)$.

Therefore the analyticity of $Y(t)$ also fails at $t = 0$. $Y(t)$ is not analytic at $t = ph,\ p\in\{0,1,2,\cdots\}$. See [11] for discussions on analytic functions.

Next we investigate the singularity of $Y(t): Y(t)$ is singular if

$e^{at} + b(t-h)e^{a(t-h)} = 0$, iff $t = \frac{1+bhe^{-ah}}{b} = \frac{1}{b} + \mathrm{h}e^{-ah}$, for some $t\in[h, 2h]$. Let us examine the feasibility of such t value.

$$h \le \frac{1}{b} + he^{-ah} \le 2h \ \text{ iff } \ \frac{1}{\left(2 - e^{-ah}\right)h} \le b \le \frac{1}{\left(1 - e^{-ah}\right)h},\ a \notin \left\{0, -\frac{\mathrm{Ln}(2)}{h}\right\},\ h \neq 0.$$

Such choice of b is certainly feasible; for example $a=1,\ h=1 \Rightarrow \frac{1}{2-e^{-1}} \le\ b \le \frac{1}{1-e^{-1}}$. Therefore in general $Y(t)$ is singular for $t>h.$ Note that $Y(t)$ is nonsingular for $t\in(0,h],$ and by construction continuous except at 0, noting that $\lim\limits_{t\to 0^-} Y(t)=0,$ since $Y(t)=\begin{cases}0, & t<0\\ 1, & t=0\end{cases}$

and $\lim\limits_{t\to 0^+} Y(t)=Y(0)=1.$

Corollary to the theorem

Let $a=0$ and $\phi(t)=\theta, \forall t\in[-h,0].$ Then

$$x(t)=\left[1+\sum_{i=0}^{k}\frac{b^{i+1}}{(i+1)!}\left(t-ih\right)^{i+1}\right]\theta,\ t\in J_k,k\ge 0. \tag{22}$$

$$s\in[-h,0]\Rightarrow -s-h\in[-h,\ 0];\ t\in J_0\Rightarrow t-s-h\in[-h,\ h];\ 0\le t-s-h\le h \text{ iff } t-2h\le s\le t-h$$

$$\Rightarrow -h\le s\le t-h, \text{for the feasibility of } s.$$

$$x(t)=Y(t)\phi(0)+b\int_{-h}^{t-h}Y(t-s-h)\phi(s)ds, t\in J_0 \tag{23}$$

$$\Rightarrow x(t)=\theta+b\int_{-h}^{t-h}\theta\, ds=\left[1+bt\right]\theta, t\in J_0 \tag{24}$$

This is consistent with (22).

The following analysis is imperative for $s\in[-h,\ 0]$ and $t\in J_1$:

$s\in[-h,0]\Rightarrow -s-h\in[-h,0];\ t\in J_1\Rightarrow t-s-h\in[0,2h];\ t-s-h\ge h$ iff $s\le t-2h.$ Thus

$t-s-h\ge h$ iff $s\in[-h,t-2h],$ and $t-s-h\le h$ iff $s\in[t-2h,0]\Rightarrow s\in[-h,0].$

$0\le t-s-h\le h\Rightarrow Y(t-s-h)=1;$

$h\le t-s-1\le 2h\Rightarrow Y(t-s-h)=1+b(t-s-2h).$ We deduce from $\phi(0)=\theta$ that

$$x(t)=[1+b(t-h)]\theta+\int_{t-2h}^{0}b\theta ds+\int_{-h}^{t-2h}b\left[1+b(t-s-2h)\right]\theta ds$$

$$=[1+b(t-h)+b(2h-t)+b(t-h)+b^2(t-2h)(t-h)-\frac{b^2}{2!}(t-2h)^2+\frac{b^2}{2!}h^2]\theta$$

$$=\left[1+bt+\frac{b^2t^2}{2!}-b^2ht+\frac{b^2}{2!}h^2\right]\theta=\left[1+bt+\frac{b^2}{2!}(t-h)^2\right]\theta.$$

Therefore the corollary is valid for $t\in J_1$. The rest of the proof is by induction.

Assume that the corollary is valid for $1\le p\le k,$ for some integer $k\ge 2.$ Then for $t\in J_{k+1},$

$s\in[-h,0]\Rightarrow t-s-h\in[kh,\ (k+2)h]=J_k\cup J_{k+1};\ t-s-h\in J_k$ iff $t-s-h\le(k+1)h$

iff $s\ge t-(k+2)h.$ Thus $t-(k+2)h\le s\le 0,$ for s-feasibility.

$t-s-h\in J_{k+1}$ iff $t-s-h\ge(k+1)h$ iff $s\le t-(k+2)h.$ Thus $-h\le s\le t-(k+2)h,$

for s - feasibility. Hence this analysis combined with expression (9) of the theorem and the variation of constants formula (18) yield the following expression for the solution:

$$x(t)=\left[1+\sum_{i=1}^{k+1}\frac{b^i}{i!}(t-ih)^i\right]\theta+\int_{t-(k+2)h}^{0}\left[b+\sum_{i=1}^{k}\frac{b^{i+1}}{i!}(t-s-[i+1]h)^i\right]\theta ds$$

$$+\int_{-h}^{t-(k+2)h}\left[b+\sum_{i=1}^{k+1}\frac{b^{i+1}}{i!}(t-s-[i+1]h)^i\right]\theta ds$$

$$\Rightarrow x(t)=\left[1+\sum_{i=1}^{k+1}\frac{b^i}{i!}(t-ih)^i\right]\theta+\left[(k+2h-t)b-\sum_{i=1}^{k}\frac{b^{i+1}}{(i+1)!}(t-[i+1]h)^{i+1}\right]\theta+\left[\sum_{i=1}^{k}\frac{b^{i+1}}{(i+1)!}([k+1-i]h)^{i+1}\right]\theta$$

$$+\left[[t-(k+1)h]b-\sum_{i=1}^{k+1}\frac{b^{i+1}}{(i+1)!}([k+1-i]h)^{i+1}\right]\theta+\left[\sum_{i=1}^{k+1}\frac{b^{i+1}}{(i+1)!}(t-ih)^{i+1}\right]\theta$$

By change of variables, $\sum_{i=1}^{k}\frac{b^{i+1}}{(i+1)!}(t-[i+1]h)^{i+1}=\sum_{i=2}^{k+1}\frac{b^i}{i!}(t-ih)^i$,

$$\Rightarrow x(t)=\left[1+b(t-h)+bh\right]\theta+\left[\sum_{i=1}^{k+1}\frac{b^{i+1}}{(i+1)!}(t-ih)^{i+1}\right]\theta=\left[1+\sum_{i=0}^{k+1}\frac{b^{i+1}}{(i+1)!}(t-ih)^{i+1}\right]\theta$$

Therefore the corollary is valid for $t\in J_{k+1}$, completing the proof.

Conclusion

This paper has established the structure of solutions matrices for scalar delay differential equations, without which the variation of constants formula would be doomed. It has also elucidated the computational procedure for initial function problems. The method of proof can be exploited to extend the results to systems with arbitrary continuous initial functions.

References

[1] C. Ukwu and E.J.D. Garba. The Structure of Solution Matrices for Certain Single Delay Systems and Application Instances, (Submitted) African Journal of Natural Sciences, Jos, 2014.

[2] E. N. Chukwu, Stability and Time-optimal control of hereditary systems. Academic Press, New York, 1992.

[3] R. Gabasov and F. Kirillova, The qualitative theory of optimal processes. Marcel Dekker Inc., New York, 1976.

[4] A. Manitius, Control Theory and Topics in Functional Analysis. Vol. 3, Int. Atomic Energy Agency, Vienna, 1978.

[5] G. Tadmore. Functional differential equations of retarded and neutral types: Analytical solutions and piecewise continuous controls. J. Differential equations, Vol. 51, No. 2, 1984, Pp. 151-181.

[6] C. Ukwu, Compactness of cores of targets for linear delay systems., J. Math. Analy. and Appl., Vol. 125, No. 2, August 1, 1987, pp. 323-330.

[7] C. Ukwu, Euclidean Controllability and Cores of Euclidean Targets for Differential difference systems Master of Science Thesis in Applied Math. with O.R. (Unpublished), North Carolina State University, Raleigh, N. C. U.S.A., 1992.

[8] C. Ukwu, An exposition on Cores and Controllability of differential difference systems, ABACUS Vol. 24, No. 2, 1996, pp. 276-285.

[9] C. Ukwu, (2013a). On Determining Matrices, Controllability and Cores of Targets of Certain Classes of Autonomous Functional Differential Systems with Software Development and Implementation. Doctor of Philosophy Thesis, UNIJOS (In progress).

[10] R. D. Driver. Ordinary and Delay Differential Equations. Applied Mathematical Sciences 20, Springer-Verlag, New York, 1977.

[11] C. Chidume. Applicable Functional Analysis. The Abdus Salam, International Centre for Theoretical Physics, Trieste, Italy, 2007.