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On Linearizing dynamical systems in 3-D using the inverse cube law

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Abstract

In the symmetry analysis of dynamical systems Ermanno-Bernoulli constants were used to reduce the Kepler problem to systems of oscillators and conservation law in both 2-D and 3-D. In this paper we utilize alternative constants obtained from Hamilton vector of the inverse cube law to reduce it to systems of oscillators and conservation law in 3-D. We compared the effectiveness of both reduction constants through their reduction variables in 3-D and note efficient reduction constant. Further we present a proposition which assert the equivalence of the group realizations of the emanating Lie symmetry groups from these two linearization procedures.

Keywords: Laplace transforms, Rieman-Liouville fractional integral, Caputo's fractional derivative, Mittag-Leffler function, Fractional differential equation, Damping.

1.0 Introduction

Symmetries of differential equations are well known to reduce it to quadrature. However to obtain these symmetries for some dynamical systems is not so easy talk less of the complete symmetry groups of same. The quest for the complete symmetry groups of dynamical system particularly in the case of the Kepler problem in the literature involves the reduction of dynamical systems into systems of oscillator(s) and conservation law Leach et al [1]. The chief of these reduction processes was the Ermanno-Bernoulli constants reduction variables Leach and Nucci [2]; but the report by Leach and Flessas [3] on obtaining the reduction variables using the Ermanno-Bernoulli constants seems not trivial, however we get it more easily here by utilizing a general procedure of Arunaye [4]. Hitherto, Arunaye and White [5] alternative reduction method proved sophisticated as constants obtained from the Hamilton vector of dynamical systems generate simpler reduction variables in 3-D motion. However, if the Lie symmetry groups of the reduced couple oscillators and conservation law for the two procedures of linearization, we show that under the group actions acting on the manifold M of the dynamical system, there are invertible mappings, and then we conclude that the two constants for linearization are equivalent.

2.0 Hamilton and Laplace-Runge-Lenz vectors of dynamical systems

The most general inverse cube law of motion in 3-dimention has the equation of motion given by

$$\ddot{\mathbf{x}} = \mu_1 r^{-4}\mathbf{x} + \mu_2 r^{-4}(\mathbf{x}\wedge\mathbf{L}), \qquad (2.1)$$

where μ_1 and μ_2 are functions of $L = |\mathbf{L}|$, symbol $\wedge$ represents vector product ; and system (2.1) possesses Lie symmetries of the inverse cube law

$$\ddot{\mathbf{x}} = Gr^{-4}\mathbf{x}. \qquad (2.2)$$

Equation (2.1) possesses Hamilton and Laplace-Runge-Lenz vectors respectively given by $\mathbf{K}$ and $\mathbf{J}$ White [6, 7]

$$\mathbf{K} = L^{-\frac{1}{2}}\dot{\mathbf{x}} - \tfrac{1}{2}\dot{L}L^{-\frac{3}{2}}\mathbf{x}, \qquad (2.3)$$

provided $$\mu_1 = \tfrac{1}{2}\mu_2(L\mu_2' - \tfrac{1}{2}\mu_2); \qquad (2.4)$$

and $$\mathbf{J} = -\hat{\mathbf{L}}\wedge\mathbf{K} = -L^{-\frac{3}{2}}[(r\dot{r} - \tfrac{1}{2}\mu_2)\dot{\mathbf{x}} - (|\dot{\mathbf{x}}|^2 - \tfrac{1}{2}\mu_2 r^{-1}\dot{r})\mathbf{x}], \qquad (2.5)$$

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where $\dot{L} = -\mu_2 r^{-2} L$;

and $\mathbf{L} \wedge \dot{\mathbf{x}} = \mathbf{x} \wedge (\dot{\mathbf{x}} \wedge \mathbf{x})$, $\quad \mathbf{L} \wedge \dot{\mathbf{x}} = -\mathbf{x} \wedge (\mathbf{x} \wedge \dot{\mathbf{x}}) = -r^2 \dot{\mathbf{x}} + r\dot{r}\mathbf{x}$. (2.6)

3.0 Linearizing dynamical systems in 3-D

We shall in the following demonstrate the general computation of Ermanno-Bernoulli constants of dynamical systems in 3-D and utilize them to reduce the dynamical system to systems of oscillators and conservation law by considering the inverse cube law equation of motion as illustrative example. By setting

$$x_1 + ix_2 = r\sin\theta e^{i\phi},$$
$$\dot{x}_1 + i\dot{x}_2 = [(r\sin\theta)^{\cdot} + i(r\dot{\phi}\sin\theta)]e^{i\phi}; \tag{3.1}$$

one obtains after some manipulations the Ermanno-Bernoulli constants

$$J_1 \pm iJ_2 = -L^{-\frac{3}{2}}[(r\dot{r} - \tfrac{1}{2}\mu_2)(r\sin\theta)^{\cdot} - (|\dot{\mathbf{x}}|^2 - \tfrac{1}{2}\mu_2 r^{-1}\dot{r})(r\sin\theta)$$
$$\pm i(r\dot{r} - \tfrac{1}{2}\mu_2)(r\dot{\phi}\sin\theta)]e^{\pm i\phi}. \tag{3.2}$$

Substituting for $|\dot{\mathbf{x}}|^2 = \dot{r}^2 + r^{-2}L^2$ in (3.2) we get

$$J_1 \pm iJ_2 = (v_1 \pm iv_1')e^{\pm i\phi};\ v_1 = L^{-\frac{3}{2}}\{r^{-1}L^2\sin\theta - (r\dot{r} - \tfrac{1}{2}\mu_2)(r\dot{\theta}\cos\theta)\}, \tag{3.3}$$
$$v_1' = -L^{-\frac{3}{2}}[(r\dot{r} - \tfrac{1}{2}\mu_2)(r\dot{\phi}\sin\theta)].$$

Using $\dot{J}_1 \pm i\dot{J}_2 = 0$ we have from equation (3.3) the oscillator

$$v_1'' + v_1 = 0. \tag{3.4}$$

We also obtain the second oscillator and the conservation law as follows. Let

$$\hat{L}_{\pm} = (\upsilon_2 \pm i\upsilon_2')e^{\pm i\phi}; \tag{3.5}$$

where $\upsilon_2 = L^{-1}r^2\dot{\phi}\sin\theta\cos\theta$, $\upsilon_2' = -L^{-1}r^2\dot{\theta}$ and, $\hat{L}_3 = L^{-1}r^2\dot{\phi}\sin^2\theta$. And υ_2 satisfies $\upsilon_2'' + \upsilon_2 = 0$. Since $(\hat{L}_3)' = 0$ (where prime denotes derivation with respect to ϕ), we have $v_2 = \upsilon_2(\hat{L}_3)^{-1}$ (also $v_2'' + v_2 = 0$) as a constant multiple of υ_2; but $v_2 = \upsilon_2(\hat{L}_3)^{-1} = \cot\theta$ produces the second oscillator ($v_2'' + v_2 = 0$) for the reduced system. We have thus proved that the second reduction variable $v_2 = \cot\theta$ is true for all dynamical systems in three-dimensions. The third variable that produce the conservation law is arguably obtained Arunaye [7] as in the second-dimensional case as $v_3 = W(L) - \beta$ where β depends on both θ and ϕ [the resultant angle of motion on the plane of motion. i.e., β is the angle between the plane perpendicular to the plane of motion and the plane of motion], which satisfies $v_3' = 0$. Thus the reduced system is

$$v_1'' + v_1 = 0,$$
$$v_2'' + v_2 = 0, \tag{3.6}$$
$$v_3' = 0.$$

4.0 Alternative constants for linearizing dynamical systems

In the following, we demonstrate that the alternative constants which is obtain from $\mathbf{K}$, which is also used for linearizing dynamical systems in three dimensions could be computed using the inverse cube law equation of motion as illustration . The most general Hamilton vector $\mathbf{K}$ for system (2.1) is given by (2.3). By substituting (3.1) into (2.3) we obtain after some manipulations an alternative to Ermanno-Bernoulli constants as

$$K_1 + iK_2 = L^{-\frac{1}{2}}[(r\sin\theta)^{\cdot} + i(r\dot{\phi}\sin\theta) - \tfrac{1}{2}\dot{L}L^{-\frac{3}{2}}(r\sin\theta)]e^{i\phi},$$
$$= L^{-\frac{1}{2}}[(r\sin\theta)^{\cdot} + \tfrac{1}{2r}\mu_2\sin\theta + i(r\dot{\phi}\sin\theta)]e^{i\phi}; \tag{4.1}$$

i.e. $$K_1 \pm iK_2 = (u_1 \pm iu_1')e^{\pm i\phi};\ u_1 = (L^{-\frac{1}{2}}r\sin\theta)^{\cdot};\ u_1' = L^{-\frac{1}{2}}\dot{\phi}r\sin\theta. \tag{4.2}$$

Thus (2.1) in three-dimensions is reduced to

$$u_1'' + u_1 = 0\ ,\quad u_2'' + u_2 = 0\ ,\quad u_3' = 0\ , \tag{4.3}$$

where $u_1 = (L^{-\frac{1}{2}} r\sin\theta)^{\cdot}$; $u_1' = L^{-\frac{1}{2}}\dot{\phi} r\sin\theta$ and $u_2 = v_2$ $u_3 = v_3$ are as in (3.6) above. The variables u_2 and u_3 can also be obtained as in Leach et al. [1], Leach and Nucci [2]. Systems (3.6) and (4.3) each have sixteen Lie symmetries [1, 2, 3].

5.0 The group representation

Proposition *The group transformations for the Ermanno-Bernoulli and the alternative constants* $J_{\pm} = (u \pm iu')e^{\pm i\theta}$ *and* $K_{\pm} = (v' \pm iv)e^{\pm i\theta}$ *respectively are equivalent in their group representations.*

Let $T^g, U^g \in M$ be transformation groups on the manifold M. And let $f : M \to M$ be such that f^{-1} is a diffeomorphism. We define the following:

$$x = (u,\theta)\ ,\ \bar{x} = T^g(x)\ ,\ f(x) = x' \text{ and, } g = e^{\lambda A},\ g \in SL(3),\ A = sl(3).$$

Where $g \in SL(3)$ the Lie symmetry is group, and $A = sl(3)$ is the corresponding algebra.

So, $f^{-1}(\bar{x}') = T^g f^{-1}(x')$, $\bar{x}' = fT^g f^{-1}(x') = U^g(x')$.

Similarly, $y = (v,\theta)$, $\bar{y} = T^h(y)$ where $v = u'$, $h = e^{\lambda B}$, $h \in SL(3)$, $B = sl(3)$.

So, $y = (v,\theta) = f(u,\theta)$ and $f(\bar{x}) = T^h f(x)$; $f^4(y) = u$, $f^2(y) = -u$ and $f^4 = 1$.

So for $\bar{x} = f^{-1}T^h f(x) = T^g(x)$, one obtains equivalent group transformations T^g, $U^g \in M$.

So that the map $W = f_* v \ \Rightarrow\ W_y = f_*(v_x)$; where $v = r^{-1}$.

Consider the function $v = f(u) = \alpha u + \beta u'$.

We have $u'' = \alpha(\alpha u + \beta u') + \beta(\alpha u' + \beta u'')$,

and

$$(f - \alpha)(u) = \beta v',$$
$$(f - \alpha)^2(u) = -\beta^2 u,$$
$$\{f^2 - 2\alpha f + (\alpha^2 + \beta^2)\}(u) = 0,$$
$$f(f - 2\alpha)(u) = -(\alpha^2 + \beta^2)(u) \ \Rightarrow\ f(2\alpha - f) = (\alpha^2 + \beta^2).$$

i.e. $f\left(\frac{2\alpha - f}{\alpha^2 + \beta^2}\right) = 1 \ \Rightarrow\ ff^{-1} = 1$, $\left(\frac{2\alpha - f}{\alpha^2 + \beta^2}\right) = f^{-1}$.

Showing that f is invertible.

Now one considers the group $W^g \in M$, on the manifold of solution, the following group actions endeared:

$$T^g \begin{pmatrix} u_1 \\ \phi \\ u_2 \\ v \end{pmatrix} = \begin{pmatrix} \bar{u}_1 \\ \bar{\phi} \\ \bar{u}_2 \\ \bar{v} \end{pmatrix}, \qquad W^g \begin{pmatrix} u_1' \\ \phi \\ u_2 \\ v \end{pmatrix} = \begin{pmatrix} \bar{u}_1{}' \\ \bar{\phi} \\ \bar{u}_2 \\ \bar{v} \end{pmatrix},$$

$$f \begin{pmatrix} u_1 \\ \phi \\ u_2 \\ v \end{pmatrix} = \begin{pmatrix} u_1' \\ \phi \\ u_2 \\ v \end{pmatrix}, \qquad W^g f \begin{pmatrix} u_1 \\ \phi \\ u_2 \\ v \end{pmatrix} = f \begin{pmatrix} \bar{u}_1 \\ \bar{\phi} \\ \bar{u}_2 \\ \bar{v} \end{pmatrix},$$

$$f^{-1} W^g f \begin{pmatrix} u_1 \\ \phi \\ u_2 \\ v \end{pmatrix} = \begin{pmatrix} \bar{u}_1 \\ \bar{\phi} \\ \bar{u}_2 \\ \bar{v} \end{pmatrix} = T^g \begin{pmatrix} u_1 \\ \phi \\ u_2 \\ v \end{pmatrix};$$

$$U\begin{pmatrix} r \\ \phi \\ L \\ \nu \end{pmatrix} = \begin{pmatrix} u_1 \\ \phi \\ u_2 \\ \nu \end{pmatrix}, \ \nu = \dot{r}.$$

So, $f^{-1}W^g fU = UT^g$

and $U^{-1}f^{-1}W^g fU = T^g \ \Rightarrow \ (fU)^{-1}W^g(fU) = T^g.$ (5.1)

Also $A^g\begin{pmatrix} r \\ \phi \\ L \\ \nu \end{pmatrix} = \begin{pmatrix} \bar{r} \\ \bar{\phi} \\ \bar{L} \\ \bar{\nu} \end{pmatrix}, \ B^g\begin{pmatrix} r' \\ \phi \\ L \\ \nu \end{pmatrix} = \begin{pmatrix} \bar{r}' \\ \bar{\phi} \\ \bar{L} \\ \bar{\nu} \end{pmatrix}$ and $f\begin{pmatrix} r \\ \phi \\ L \\ \nu \end{pmatrix} = \begin{pmatrix} r' \\ \phi \\ L \\ \nu \end{pmatrix},$

$$B^g f\begin{pmatrix} r \\ \phi \\ L \\ \nu \end{pmatrix} = B^g\begin{pmatrix} r' \\ \phi \\ L \\ \nu \end{pmatrix} = \begin{pmatrix} \bar{r}' \\ \bar{\phi} \\ \bar{L} \\ \bar{\nu} \end{pmatrix} = f\begin{pmatrix} \bar{r} \\ \bar{\phi} \\ \bar{L} \\ \bar{\nu} \end{pmatrix}.$$

So that $f^{-1}B^g f\begin{pmatrix} r \\ \phi \\ L \\ \nu \end{pmatrix} = A^g\begin{pmatrix} r \\ \phi \\ L \\ \nu \end{pmatrix}$. i.e. $f^{-1}B^g f = A^g$ (5.2)

From (5.1) and (5.2) we have shown that the transformations which map solutions to solutions in the solution manifold are equivalent in their group realization.

6.0 Concluding remarks

Comparing equations (4.2) and (3.3) we observed that the natural reduction variables from the alternative constants seem less complicated than those obtained from Ermanno-Bernoulli constants in 3-D systems. We have also shown that the group representation of the Lie symmetry of system of oscillators and conservation law emanating from these two reduction constants for reducing dynamical systems are equivalent. Hence the proposition follows and we assert that the Ermanno-Bernoulli constants are sufficient for reducing dynamical systems to coupled harmonic oscillators and conservation law.

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