Journal of the Nigerian Association of Mathematical Physics
Volume 26, (March, 2014), pp 8 – 11

The Structure of a Completely Simple Semigroup S^* "Particularly Nice"

[1]Ezeako Lawrence N., [1]Eze Everestus Obinwanne and [2]Ugbene I. J.

[1]Department of Mathematics,
Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria.
[2]Department of Mathematics/Statistics,
Caritas University Amorji-Nike, Enugu, State, Nigeria.

Abstract

A completely Simple Semigroup S^* is a disjoint union of its maximal subgroups. It has only one D – class (since it is Bisimple and Regular). In S^*, each H – class is a group and the concept of L – class [R - class] principal left [right] ideal and maximal left [right] ideal, coincide. This paper examines the structure of S^* and proves all the theorems stated herein, as inherent characteristics attributable to any completely simple semigroup.

Keywords: Green's Relations; L, R, D, H [1], Maximal subgroup, Principal left [right] ideal, Minimal left [right] ideal, Minimal left [right] ideal, Simple semigroup, Primitive idempotent.
2010 Mathematics Subject Classification: 20M07, 20M12, 20M17

Preliminaries

1. An element a of semigroup S is called regular if there is an $x \in S$ such that $a = axa$. A D – class D, is called regular if every element of D is regular.
2. An element of a semigroup is regular $\Leftrightarrow Ra\ [La]$ contains an idempotent. (See [1] for proof). Thus;
3. If a D – class D of a semigroup S contains a regular element; all elements of D are regular
4. If a D – class D of a semigroup S is regular, then every L – class [R - class] contained in a D contains an idempotent.
5. An idempotent e of a semigroup S is a right identity element for Le, a left identity for Re, and an identity for He. (See [1], for proof). Thus:
6. No H – class contains more than one idempotent.
7. (J. A. Green) If a, b and ab, belong to the same H – class H of a semigroup S, then H is a subgroup of S. (See [2], for proof).
8. If a and b are elements of a semigroup S, then $ab \in Ra \cap Lb \Leftrightarrow Rb \cap La$ contains an idempotent. If this is so, then $aHb = HaHb = Hab = Ra \cap Lb$ (See [2] for proof)
9. Let E be the set of idempotents of a semigroup S. If $e, f \in E$ define $e \leq f$ if $ef = fe = e$. Thus;
10. If S has a zero, then $0 \leq e\ \forall\ e \in E$.
11. An idempotent f is called primitive if $e \leq f$ for $e, f \in E$ implies $e = f$.
12. A completely Simple Semigroup is a simple semigroup containing a primitive idempotent.
13. A subgroup of a semigroup S is a maximal subgroup of S if it is not properly contained in any other subgroup of S
14. A semigroup need not contain a subgroup. A semigroup will contain a subgroup if and only if it contains an idempotent.
15. If S is a semigroup with identity e, the group of units of eSe is denoted by $H(e)$.
16. $H(e)$ contains every subgroup G of S that meets $H(e)$, that is, if $G \cap H(e) \neq \varphi$, then $G \subseteq H(e)$.

Corresponding author: ***Eze, Everestus Obinwanne,*** E-mail: obinwanneeze@gmail.com, Tel.: +2348083964662

Proof:

Suppose that $G \cap H(e) \neq \varphi$, then $\exists\, x \in S \ni x \in G \cap H(e)$, i.e. $x, x^{-1} \in G$ and $x, x^{-1} \in H(e)$. So $x = 1x = x1, 1 \in G$, 1 being the identity in G, and $x = ex = xe, e \in H(e), e$ being the identity in $H(e). e = 1$ since $xx^{-1} = e = 1$.

Similarly, $y^{-1} = ey^{-1}e \in eSe. (eye)(ey^{-1}e) = (ey)(ee)(y^{-1}e) = (ey)e(y^{-1}e) = (ey)1(y^{-1}e) = e(yy^{-1})e = e1e = eee = e \in H(e)$.

So $y \in H(e), i.e. y \in G \Rightarrow y \in H(e)$. Hence, G $\subseteq H(e)$. Thus:

17. The groups $H(e), e \in S$, are just the maximal subgroup of S.
18. For the study of the structure of completely simple semigroups S^*, we tacitly assume that the semigroup S^* has no zero element, 0. Then we consider a minimal left[right] ideal of S^* or the minimal ideal of S^*, we know that these objects are not trivial.
19. Let $a \in S$, so that $\{a\} \subseteq S$, then $a \cup Sa[a \cup aS]$ is the principal left[right] ideal generated by a.

Theorems

Theorem 1: The maximal subgroups of any semigroup S are the H - classes of S containing idempotents.

Proof:

Let $e \in S$ be an idempotent. $e \in He$ and $e \in H(e) \therefore He \cap H(e) \neq \varphi$. So by (item 16 of the preliminaries above),

$$He \subseteq H(e) \qquad (i)$$

Let $b \in H(e)$ where $H(e)$ where $H(e)$ is a group with identity e,

$b = eb \in eS = be \in Se$, So bRe and $bLe \Rightarrow b \in He$. That is $b \in H(e) \Rightarrow b \in H(e)$

$$\therefore H(e) \subseteq He \qquad (ii)$$

Equations $(i) and\ (ii) \Rightarrow He = H(e)$

Theorem 2: If L is a minima left [right] ideal of a semigroup S, then $L = Sa[aS]$, For all $a \in L$. Hence L is an L – class of S.

Proof: Trivial. For $SL \subseteq L$ and for every $a \in L, Sa \subseteq L$. Sa is a left ideal and L is a minimal left ideal $\Rightarrow L = Sa$. Let $a, b \in L$. Then $a \cup Sa = L = b \cup Sb$. So all elements of L are in the same L – class, i.e. $L \subseteq La$, say. Let $c \in La$ then $c \in L$. So L is an L – class of S.

Theorem 3: Let S be a simple semigroup having a minimal left ideal L, and a minimal right ideal R. Then

a. $LR = S$
b. $RL = R \cap L$, and
c. RL is a group.

If e is the identity of $R \cap L$, then $R = eS, R \cap L = eSe$, and e is a primitive idempotent.

Proof:

a. $S(LR) = (SL)R \subseteq LR, (LR)S = L(RS) \subseteq LR$. Hence LR is a left ideal and a right ideal of S. So LR is an ideal of S. S is simple$\Rightarrow LR = S$.
b. $R, L \subseteq S$. R right ideal $\Rightarrow RS \subseteq R \Rightarrow RL \subseteq R$
   L, left ideal $SL \subseteq L \Rightarrow RL \subseteq L. So\ RL \subseteq L \cap R \qquad (iii)$
   Let $x \in L \cap R$, then $x \in L$ and $x \in R$. Let $a \in L$ and let $b \in R$. Then $xLa\ and\ xRb, \therefore aDb$ $\Rightarrow$ S is left simple and S is right simple $\Rightarrow$ S has one L – class and S has one R – class
   i.e.$x \in L \cap R \Rightarrow x \in RL, i.e. L \cap R \subseteq RL \qquad (iv)$
   From (iii) and (iv) $RL = L \cap R$■
c. To show that RL is a group, it suffices to show that RL is a left simple and a right simple subsemigroup of S. Suppose that $s_1 \in RL$ then $s_1 = r_1 l_1$ where $r_1 \in R, l_1 \in L$. But from Theorem 2, $R = s_1 S, L = Ss_1$.

From part (a)$S = LR, S \neq \varphi. So\ LR = Ls_1, R \neq \varphi \Rightarrow Ls_1 \neq \varphi, S(Ls_1) = (SL)s_1 \subseteq Ls_1 \subseteq L \Rightarrow Ls_1$ is a left ideal. By the minimality of L in S, L$s_1 \subseteq L \Rightarrow Ls_1 = L,$

$$\therefore RLs_1 = RL \qquad (v)$$

Similarly s_1R is a right ideal of S and $s_1R = R$ by the minimality of R in S.

$$s_1RL = RL \qquad (vi)$$

In general, $SS \subseteq S$. But S is simple $\Rightarrow SS = S$. If e is the identity of the group $R \cap L = RL$, then $e \in R$ and $e \in L, R = aS$, for every $a \in R, \Rightarrow R = es. L = Sa$ for every $a \in L \Rightarrow L = Se. R \cap L = RL = (eS)(Se) = eSe$. Since $e \subseteq R \cap L$ is the identity $\Rightarrow$ e is unique $\Rightarrow$ e is a primitive idempotent■

Theorem 4: Let S^* be completely simple semigroup. Let e bea primitive idempotent of S^*. Then $S^*e = [eS^*]$ is a minimal left [right] ideal of S^* and S^*e is a group having e as identity.

Proof: Let $A \subseteq eS^*$ be a right of S^*, contained in eS^*. We have to show that $A = eS^*$. So let $a \in A$ then $a = es$ for some $s \in S^*, ea = ees = es = a \in eS^*, so\ A \subseteq eS^*$. Since S^* is simple , $S^*aS^* = S$. For every $s \in S^*, es \in S^*\ and\ es = a \in eS^* \Rightarrow eS^* \subseteq A. So\ A = eS^*$ is a minimal right ideal. Similarly S^*e is a minimal left ideal. Hence from Theorem 3,$\Rightarrow eS^*e$ is a group having e as identity.■

Comment 1: Theorems 3 & 4 can be summarised as follows: A simple Semigroup S, becomes Completely Simple if and only if S contains at least one minimal left ideal and one minimal right ideal.

Theorem 5: Let M be a minimal left ideal of a semigroup S. Let $s \in S$. Then M is a minimal left ideal of S.

Proof: Trivial. $S(Ms) = (SM)s \subseteq Ms \Rightarrow Ms$ is also a left ideal of S. Ms is a minimal left ideal of S by the minimality of M in S.

Theorem 6: A completely simple semigroup S^* is the union of its minimal left [right] ideals.

Proof: $S^*$is completely simple $\Rightarrow S^*$ has a minimal left ideal L, say, and a minimal right ideal R, by Theorem 4. By Theorem 3, $LR = S^*$. By Theorem 5, $Lr(r \in R)$ is also a minimal left ideal of S^*. By Theorem 2, Lr is an L – class of S^*. Now $S^* = LR = \bigcup_{reR} Lr$ so S^* is the union of the minimal left ideals Lr, and this is all of them. Similarly S^* is the union of the minimal right ideals 1R with $1 \in L$.

Comment 2: By Theorem 2, the union of the minimal left [right] ideals that make up S^*, is a disjoin union of minimal left [right] ideals of S^*.

Theorem 7: A completely simple semigroup S^* is Bisimple and Regular.

Proof: $S^*$is completely simple $\Rightarrow S^* = \cup L = \cup R$, where the L's are its minimal left ideals and the R's are its minimal right ideals. By Theorem 2, the L's are L – classes of S^* and the R's are the R – classes of S^* and the R's are the R – classes of S^*. By Theorem 3, $L \cap R$ is a group. Hence, $L \cap R \neq \varphi$. For any L – class L, and for any R – class R of S^*, if D is a D – class of S^*, then $L \cup R \subseteq D$, for any L and for any R. So S^* has a single D – class. Hence S^* is Bisimple.■

Also S^* is completely simple $\Rightarrow S^*$ has a primitive idempotent e, say e is regular in S^* since $e = eee, e \in S^*$. Since S^* consists of a single D – class, D, with a regular element e, $\Rightarrow$ all elements of D are regular (by item 3 of Preliminaries). Hence S^* is Regular. (See [3] also, for proof)

Theorem 8: If S^* is a completely simple semigroup, each L – class [R - class] is a minimal left [right] ideal of S^*.

Proof: Trivial. By Theorem 6, S^* is the disjoint union of its minimal left [right] ideal. Always, a minimal left [right] ideal of a semigroup is an L – class [R - class] of the semigroup.

Theorem 9: Let S^* be a completely simple semigroup. For each $a \in S^*$, Ha is a group.

Proof: For each $a \in S^*$, let La be the L – class of a, let Ra be the R – class of a. By theorem 8, La is a minimal left ideal; Ra is a minimal right ideal. By Theorem 3, $RaLa = Ra \cap La = Ha$, which is a group.

Theorem 10: Let S^* be a completely simple semigroup. For each $a, b \in S^*, ab \in Ra \cap Lb = HaHb = Hab$.

Proof: Ra is a minimal right ideal, Lb is a minimal left ideal. $ab \in RaLb = Ra \cap Lb$ which is a group, by Theorem 3, i.e. $ab \in Ra \cap Lb$. Hence, (by item 8 of the Preliminaries) $Ra \cap Lb = HaHb = Hab$.

Comment 3: Theorem 1 & Theorem 9 confirm that S^* is a disjoint union of its maximal subgroups. Theorem 7 confirms that S^* has only one D – class.

Theorem 11: A left[right] ideal of S^* is a principal left[right] ideal if and only if it is minimal.

Proof: (⇒)Suppose that La is a principal left ideal generated by $a \in S^*$, then $La = a \cup S^*a$. By Theorem 8, La is minimal.

(⇐) Suppose that La is a minimal left ideal of S^*, then for $a \in La, La = S^*a = a \cup S^*a$ (By Theorem 2) i.e. La is a principal left of S^*.

Comment 4: Theorem 2 & Theorem 11 confirm that in S^*, the concept of L – class[R - class], principal left[right] ideals, and minimal left[right] ideals, coincide. Of course these concepts differ (or they are not coincident) in an arbitrary semigroup [4]

Observations/Conclusion

Indeed comments 1, 2, 3, 4, confirm that the structure of a completelysimple semigroup S^* is “Particurlarly nice”.

References

[1]Howie, J. M., 1976, *An Introduction to Semigroup Theory*,Academic Press, London.

[2]Clifford, A. H. And Preston G. B.,1961 and 1967,*The Algebraic Theory of Semigroups*,Amer. Math. Soc., Vol. 1 & 2, Providence, R. I.

[3] Nambooripad, K. S. S., 1975, *Structure of Regular Semigroups 2:the general case*, Semigroup Forum 9, p 364 – 371.

[4] Petrich, M., 1977, *Lectures in Semigroups*, Wiley, London.