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On Spacetime Singularities and the Liouville Black Hole

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Abstract

A class of exact solutions of the Einstein field equations due to Robinson and Trautman is reduced to a Liouville equation when the corresponding Weyl tensor is of type N. Singularities are discussed in the context of the initial value problem based on Kovalevskaya analysis in a Lobaschevsky plane. Only those singularities which develop from regular initial configurations are considered. The solutions are found to be non singular on only one copy of the Lobaschevsky plane and give the Liouville black holes under certain conditions.

1.0 Introduction

The issue of space-time singularities arose very early in the history of general relativity. There however has arisen some form of confusion from the onset as to the existence of singularities and their interpretations. A useful source of information on the confusion surrounding the subject in the first fifty years of general relativity can be found in Earman [1] as well as an article by Kennefick [2], which describes the early difficulties encountered by Einstein himself. A survey of the understanding that has been made so far with regard to space-time singularities is presented in this section.

Before concentrating on general relativity, it is useful to think more generally about the concept of a singularity in a physical theory. In the following, the emphasis is on classical field theories although some of the discussion may be of relevance to quantum theory as well. When a physical system is modeled within a classical field theory, solutions of the field equations are considered. If it happens that physically relevant quantities become infinite at some point of space then we say that there is a singularity. Since the physical theory ceases to make sense when basic quantities become infinite a singularity is a sign that the theory has been applied beyond its domain of validity. To get a better description a theory of wider applicability should be used. Note that the occurrence of singularities does not say that a theory is bad; it only sets limits on the domain of physical phenomena where it can be applied.

In fact almost any field theory allows solutions with singularities if attention is not restricted to those solutions which are likely to be physically relevant. In this context a useful criterion is provided by the specification of solutions by initial data. This means that we only consider solutions which have the property that there is some time at which they contain no singularities. Then any singularities which occur must be the result of a dynamical evolution. With this motivation, singularities will be discussed in the context of the initial value problem. Only those singularities are considered which develop from regular initial configurations. This has the consequence that linear field theories, such as source-free Maxwell theory, are free of singularities. In the case of the Einstein equations, the basic equations of general relativity, the notion of singularity becomes more complicated due to the following fact. A solution of the Einstein equations consists not just of the spacetime metric, which describes the gravitational field and the geometry of space-time, but also the spacetime manifold on which the metric is defined. In the case of a field theory in Newtonian physics or special relativity we can say that a solution becomes singular at certain points of space-time, where the basic physical quantities are not defined. Each of these points can be called a singularity. On the other hand, a singularity in general relativity cannot be a point of space-time, since by definition the space-time structure would not be defined there.

In general relativity the wordline of a free particle is described by a curve in space-time which is a timelike or null geodesic, for fermions or bosons respectively. There is also a natural class of time parameters along such a geodesic which, in the timelike case, coincide up to a choice of origin and a rescaling with the proper time in the rest frame of the particle. If the worldline of a particle only exists for a finite time then clearly something has gone seriously wrong. Mathematically this is called geodesic incompleteness. A spacetime which is a solution of the Einstein equations is said to be singular if it is timelike or null geodesically incomplete. Informally we say in this case that the spacetime ``contains a singularity'' but the

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definition does not include a description of what a ``singularity" or ``singular point" is. There have been attempts to define ideal points which could be added to spacetime to define a mathematical boundary representing singularities but these have had limited success (see [3], for a complete discussion). When working practically with solutions of the Einstein equations it is necessary to choose coordinates or other similar auxiliary objects in order to have a concrete description. In general relativity we are free to use any coordinate system and this leads to a problem when considering singularities. Suppose that a metric written in coordinates is such that the components of the metric become infinite as certain values of the coordinates are approached. This could be a sign that there is a spacetime singularity but it could also simply mean that those coordinates break down at some points of a perfectly regular solution. This might be confirmed by transforming to new coordinates where the metric components have a regular extension through the apparent singularities. A way of detecting singularities within a coordinate system is to find that curvature invariants become infinite. These are scalar quantities which measure the curvature of spacetime and if they become infinite this is a sure sign that a region of spacetime cannot be extended. It is still not completely clear what is happening since the singular values of the coordinates might correspond to singular behaviour in the sense of geodesic incompleteness or they might be infinitely far away.

A breakthrough in the understanding of spacetime singularities was the singularity theorem given by Penrose in 1965 as referred in [4], which identified general conditions under which a spacetime must be geodesically incomplete. This was then generalized to other situations by Hawking and others, [5 - 7]. The singularity theorems are proved by contradiction. Their strength is that the hypotheses required are very general and their weakness is that they give very little information about what actually happens dynamically. If the wordline of a particle ceases to exist after finite proper time then it is reasonable to ask for an explanation, why the particle ceased to exist. It is to be expected that some extreme physical conditions play a role. For instance, the matter density or the curvature, representing tidal forces acting on the particle, becomes large. From this point of view one would like to know that curvature invariants become unbounded along the incomplete timelike or null geodesics. The singularity theorems give no information on this question which is that of the nature of spacetime singularities. A key question about singularities in general relativity is whether they are a disaster for the theory. If a singularity can be formed and then influence the evolution of spacetime then this means a breakdown of predictability for the theory. For we cannot (at least within the classical theory) predict anything about the influence a singularity will have. A singularity which can causally influence parts of spacetime is called a naked singularity. It is important for the predictive power of general relativity that naked singularities be ruled out. This has been formulated more precisely by Penrose as the cosmic censorship hypothesis [8].

2.0 Radiative Space times

The general solution of Einstein equations in vacuum, for space admitting congruence of null geodesics that are shear free and twist free is given in terms of standard coordinates u, r, ξ, ξ^* by the Robinson –Trautman metric [9]

$$ds^2 = 2dudr + 2Hdu^2 - 2r^2 P^{-2} d\xi d\xi^* \quad (1)$$

where u and r are real and ξ is complex. The function *P* is independent of *r*. The coefficient *H* in the metric (1) is given in terms of *P* as

$$H = P^2 (\log P)_{,\xi\xi^*} - r(\log P)_{,u} - \frac{m(u)}{r} \quad (2)$$

The function *P* satisfies the fourth order equation

$$P^2\left(P^2 (\log P)_{,\xi\xi^*}\right)_{,\xi\xi^*} + 3m(\ln P)_{,u} - m_{,u} \quad (3)$$

Equation (3) is often referred to as the Robinson-Trautman equation. By coordinate freedom, m can be transformed to the value ± 1 or 0. The Gaussian curvature K of the surfaces of constant u and *r* is given by $\mathrm{K} = 2P^2 (\log P)_{,\xi\xi^*}$. In the special case when $m(u) = 0$, .the Gaussian curvature is characterized by the fact that K is a solution of the Laplace equation, so K must be either singular or constant. *P* satisfies the equation

$$2P^2 (\ln P)_{,\xi\xi^*} = -3(f + f^*) \quad (4)$$

where $f = f(u, \xi)$ is an arbitrary function of the complex variable ξ and real *u*. In the special case where *f* does not depend on *u* , equation (4) reduces to the equation

$$2P^2 (\ln P)_{,\xi\xi^*} = -3(\xi + \xi^*) \quad (5)$$

where we have made the substitution

$$\xi \to f, \qquad P \to \left|\frac{\partial f}{\partial \xi}\right| \tag{6}$$

However if *f* is real and depend on *u* only (3) reduces to the Liouville equation whose general solution has the form [10],

$$P = (\varphi + \varphi^*)\sqrt{-3f} \Big/ \left|\frac{\partial \varphi}{\partial \xi}\right| \tag{7}$$

where φ need not be analytic everywhere in the domain D; rather it can admit a distribution of isolated simple pole singularities and still produce a solution of the Liouville equation. To see this we observe from (7) that any simple pole of φ at a point ξ_0 would produce a double pole of $\frac{\partial \varphi}{\partial \xi}$ at ξ_0. However, the denominator would conspire with the numerator to produce a non zero non singular value for *P*. Another property of the solution is that the derivative $\frac{\partial \varphi}{\partial \xi}$ must not vanish anywhere in D, Since φ is holonomic we can study solutions with a complex topology of a 2 dimensional space.

3.0 Asymptotic behavior of the solutions

Let $\breve{\mathrm{H}} = \{W \in \mathbb{C} : W = a + ib, \quad b > 0\}$ be the upper half plane model of the Lobachevsky plane with the hyoerbolic metric

$$ds^2 = \frac{da^2 + db^2}{b^2} \tag{8}$$

There is a unique complex analytic covering $J : \breve{\mathrm{H}} \to X$, ramified over the marked points p_j with ramification indices k_i such that the group of deck transformations is isomorphic, up to a conjugation in PSL(2,$\mathbb{C}$) to the Fuchsian group Γ of the first kind satisfying $X \approx \Gamma \big/ \breve{\mathrm{H}}$. The pullback of the hyperbolic metric on $\bar{\mathrm{H}}$ by J^{-1} is well-defined and is a hyperbolic metric $e^{\varphi}[dz]^2$ on *X* where

$$e^{\varphi(z,z^*)} = \frac{\left|\frac{\partial J^{-1}}{\partial z}\right|^2}{\left(\operatorname{Im} J^{-1}\right)^2} \tag{9}$$

and *z* is a local complex coordinate on *X*. It satisfies the Liouville equation on $X \setminus \{p_{1,} \ldots, p_n\}$ with Gaussian curvature =-1 and has the following asymptotic behavior near the vertical points

$$e^{\varphi} \approx \frac{c_i}{|z|^{2\alpha_j}} \qquad \alpha_j = 1 - \frac{1}{k_i} \qquad \text{and} \quad z(p_i) = 0 \tag{10}$$

For the case $\alpha = 1$, the factor $|z|^{-2\alpha}$ is replaced by $\left(|z|^2 \log^2 |z|\right)^{-1}$.

3.0 Liouville black –hole solutions.

The Lowville equation on $X \setminus \{p_{1,} \ldots, p_n\}$ with asymptotics (10) has a unique solution which is given by the Liouville formula (9) with arbitrary holomorphic function J^{-1}.

The circumstance under which the spacetime associated with the Liouville solutions correspond to black-holes results from the following

(i) The spacetime is asymptotically flat
(ii) There exist an event horizon for a finite value of z
(iii) The metric signature is (-,+) for large z

Asymptotically flat Robinson-Trautman metrics coupled with their large *u* behavior is of great interest and there are only few known asymptotically flat solutions. A spacetime can be thought of being asymptotically flat if the curvature scalar R vanishes at an appropriate rate as infinity is approached along the future-directed null geodesics of the null hypersurfaces.

The condition that R vanishes can be relaxed to allow R to approach a constant if one wishes to consider cosmological black holes. The metric corresponding to (7) is

$$ds^2 = 2du(du + dr) - r^2 \frac{d\varphi d\varphi^*}{(\varphi + \varphi^*)^2} \tag{11}$$

A solution of the Liouville equation with singularities can be written as

$e^{\phi} \approx \frac{1}{r_{ii}^2 \log^2 r_i}$ as $r_i = [z - z_i] \to 0$ which has the singular behavior near some simple analytic closed curve C, so that

$$e^{\phi} \approx \frac{-4S'(z_0)}{\left(S'(z_0)(z - z_0) - (\bar{z} - \bar{z}_0)\right)^2} \qquad \text{as } z \to z_0 \in C \quad (12)$$

where S is the Schwarz function of the contour *C*: $C = \{z \in \mathbb{C} : \bar{z} = S(z)\}$ work on this is in progress for a general case for the curve *C*. by Peter Zograf. V. I. Smirnov thesis gives a complete solution for-the case of 4 singular points [11]

We see from the asymptotic behavior of solutions and the metric (11) that condition (i) is satisfied as the Ricci scalar vanishes as $\xi \to \infty$. Also the spacetime is singular at the origin $\xi = 0$ from (12) therefore there exist an event horizon along *C*. Finally, for the condition (iii), we observe that under the transformation $\int \sqrt{f(u)}du \to u,\ r/\sqrt{f(u)} \to r$ the coordinate r can be regarded as the cosmological time. The metric (11) is then expressed in terms of an arbitrary holomorphic function $\varphi(u, \xi)$ with multi-sheeted Riemann surfaces, whose solutions belong to a complex topology of two-dimensional space,

The general case (5) has only one known solution to this day found by Robinson and Trautman in 1962 [9]

$$P = \left(\xi + \bar{\xi}\right)^{\frac{3}{2}} \tag{12}$$

Equation (5) does not realistically describe gravitational waves.. Work is currently in progress to find other non trivial exact solutions to (5). See [12] for details.

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