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Kovalevskaya Analysis and the integrability of Crowdy Equation

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Abstract

Crowdy equation is reduced to the generalized Liouville equation on the Lobachevsky plane. Particular solutions are found to have a moving logarithmic branch point. A bifurcation analysis is performed which indicates that the solutions are unstable The solutions obtained in this paper corresponds to steady vortical equilibria on an axially symmetric surface which is topologically equivalent to a sphere but with different curvatures determined by a parameter γ .

1.0 Introduction

To incorporate curvature effects in both the structure and stability of vortex configurations in planetary scales, a number of authors have studied point vortices on spherical surfaces. This has a long history, dating back to the 19th century with Helmholtz initiating the point vortex model [1]. Perhaps the most significant of these studies is Crowdy's generalization of the planer Stuart vortex to the surface of a sphere using a conformal mapping of the spherical surface into the complex plane [2]. Ifidon [3] obtained a non linear elliptic partial differential equation (involving a parameter γ), which we shall refer to as Crowdy equation.

$$U_{xy} = c(1+xy)^{\gamma} e^{U} \tag{1}$$

Here subscripts denote partial derivative with respect to the variables and then *c* and γ are constant parameter. It is the primary purpose of this paper to examine the integrability of equation (1). We show that for non zero values of γ, equation (1) can be reduced to a generalized Liouville equation on the Lobachevsky plane. One of the important requirements for the integrability of distributed vortical systems is that the one-dimensional reductions of these systems have no moving singularities upon continuation of the independent variables in the transformation from the real axis to the complex plane. It is expected that these solutions must have integrable singularities since there are no solid boundaries present. Equation (1) can be studied by means of symmetry methods such as those of [4] and also by the dynamical system method. In the dynamical system theory, symmetries and first integrals are two fundamental structures of the ordinsry differential ewuation.

2.0 Generalized Liouville equation

Assume

$$U(x,y) = -W(x,y) - \gamma \log(1+xy) \tag{2}$$

Then (1) becomes

$$W_{xy} + \gamma(1+xy)^{-2} = -ce^{-W} \tag{3}$$

Let

$$W = 2\log p \tag{4}$$

where *p* is an arbitrary function of *x* and *y*. Substituting (4) in (3) gives

$$p^2(\log p^2)_{xy} + \gamma(1+xy)^{-2} p^2 = -c \tag{5}$$

Following the substitution

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$$p = (1 + xy)q(x, y) \tag{6}$$

Equation (5) becomes

$$2(1 + xy)^2 q^2 (\log q)_{xy} = c(q^2 - 1) \tag{7}$$

Where we have assumed without loss of generality that $\gamma + 2 = -c$. Equation (7) can be written as

$$2\Delta(\log q) = c(1 - q^{-2}) \tag{8}$$

Where Δ is the Laplace operator in the Lobaschevsky plane realized in the upper plane y>0 with the metric

$$ds^2 = \frac{dxdy}{(1 + xy)^2} \tag{9}$$

Using the substitution

$$\varphi = -2\log q \tag{10}$$

Equation (8) becomes the generalized Liouville equation in the Lobachevsky plane given as

$$\Delta\varphi + ce^{\varphi} = c \tag{11}$$

Which is a particular form of the equation obtained by Crowdy in his generalization of the planer Stuart vortex to the surface of a sphere using a conformal mapping of the spherical surface into the complex plane [2]. This forms one of several approaches that has been used in the literature to study vortices on spherical surfaces. Other approaches of note involves using a stream function formalism with the governing equations for the planar vortex motion in spherical coordinates [5]

3.0 Kovalevskaya Analysis of Crowdy Equation

We shall consider a semi-geodesic orthogonal system. In the two-dimensional case, the squared line element is usually written in standard notations as

$$ds^2 = du^2 + B^2(u,v)dv^2 \tag{12}$$

The Gaussian curvature is determined from the formula

$$K = -\frac{1}{B}\frac{\partial^2 B}{\partial u^2} \tag{13}$$

Surfaces with constant negative Gaussian curvature (K-surfaces), provides a model for the Lobaschevsky plane. For a K-surface with metric (13), the Laplace operator in the Lobaschevsky plane takes the form

$$\Delta = \frac{\partial^2}{\partial u^2} + \sigma(u)\frac{\partial}{\partial u} + \Omega^{-2}\frac{\partial^2}{\partial v^2} \tag{14}$$

where

$$\sigma(u) = \frac{d\log\Omega}{du} \tag{15}$$

The function Ω has the value -1 corresponding to one of the canonical forms of the metric (12). Substituting $x = e^u + iv$, equation (7) becomes

$$2q\left(\frac{d^2q}{du^2} + \sigma(u)\frac{dq}{du} - 2\left(\frac{dq}{du}\right)^2\right) = c(q^2 - 1) \tag{16}$$

Equation (16) has been solved [6] for a particular choice of $\sigma(u) = \cosh u$ subject to the condition $q = 0$ at some point u_0 say for which $\sigma(u_0)$ is analytic. The solution found is not analytic and has a-moving logarithmic branch point.

$$q = \sqrt{c}\left(u - u_0 - \frac{\sigma_0(u - u_0)^2}{2} + A(u - u_0)^3 + (c - \sigma_0^2 - \sigma_1)(u - u_0)^3\frac{\log|u - u_0|}{3} + \ldots\right) \tag{17}$$

A and u_0 are arbitrary constants. The existence of logarithmic branch point is an indication of the failure of the Kovalevskaya Painlive test for the integrability of Crowdy equation for non zero values of γ. This is not conclusive and further test such as the Wahlquist Estabrook prolongation method can be performed. For $\sigma(u) = -1$, a bifurcarion analysis show the solutions corresponding to $\gamma = 1$ and $q \pm 1$ are unstable at a saddle point. A small perturbation of the solutions, leads to an integral curve which passes through the neighborhood of the saddle point. The solutions however correspond to steady vortical equilibria on an axially symmetric surface which is topologically equivalent to a sphere but with different curvatures determined by a non zero parameter γ .

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