Journal of the Nigerian Association of Mathematical Physics
Volume **25 (November, 2013), pp 509 – 520**
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A note on Ricci Flow on Closed Surfaces M^2 with $\chi(M^2) > 0$

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Abstract

We describe the process of flowing 2 closed surfaces by the Ricci flow and make some remarks when the Euler characteristic $\chi(M^2)$ of the surface is positive. Indeed, either the round 2-sphere, S^2 or its Z_2 quotient, RP^2 is the only gradient shrinking Ricci Soliton. As a by product, we obtain differential Harnack estimates on positive solutions of conjugate heat equation defined on a surface with nonnegative scalar curvature.

***Keywords*:** Ricci soliton, Monotonicity Formulae, Einstein metric, Kazdan-Warner Identities, Uniformization Theorem.

1.0 Introduction

The goal of this article is to describe how closed surfaces can be deformed using a geometric nonlinear heat-flow, called the Ricci flow. By a closed surface we mean a compact 2-manifold without boundary; if it is simply connected, then it is topologically equivalent to 2-sphere. On the other hand, the Ricci flow is a process of deforming metric tensor on a general $n-$manifold $(n \geq 2)$, [1 - 3]. The Ricci flow was first understood in dimensions higher than 2, as it provides a complete classification of three manifolds, that is, the Thurston Geometrization conjecture which has Poincaré conjecture as a special case. See for instance [4], and [5]. The Ricci flow in 2-dimension is conformal and if the total surface area should be preserved during the evolution, Ricci flow will definitely converge to a constant Gaussian curvature metric everywhere in the conformal class, that is, the limiting metric is conformal to the background metric and of course to metric $g(t)$ at any time t. This provides a proof of Uniformization Theorem of Poincaré and Koebe [6, 7, 8]. However, it is much more difficult to establish the convergence of the Ricci flow when the Euler characteristic of the surface is positive.

The outlines of this note follow; in section 2, we introduce the Ricci flow as an initial value problem for nonlinear heat equation and show how M^2 is being deformed under the normalized version of the flow. In section 3, we discuss the Ricci flow approach to the uniformization of surfaces. In section 4, we make some remarks on the existence and uniqueness of Ricci soliton with examples when the Euler characteristic is positive. In section 5, we discuss monotonicity of Hamilton's surface entropy and consequently show that the metric of round 2-sphere is a gradient shrinking soliton and lastly in section 6, we further obtain some results as by-products of Theorem (5.3) in relation to differential Harnack's estimates on positive solutions of conjugate heat equation on the surfaces. For the purpose of this section we refer to the standard references [9-11] and some related works [12 - 15] on Kahler-Ricci flow, Gaussian curvature flow and mean curvature flow.

2.0 The Ricci Flow

Let us consider an initial value problem for nonlinear heat equation (2.1) introduced in [1], defined on an n-dimensional manifold M, $n \geq 2$, together with a one-parameter family of Riemannian metric $g(t), t \in (-\infty, \infty)$

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij} \\ g(0) = g_0 \end{cases} \tag{2.1}$$

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where R_{ij} is the Ricci curvature tensor. The equation evolves and smoothens out the metric of the manifold with respect to time. In particular, it deforms the metric by the negative of its curvature to give a better form of the metric. We therefore refer to (2.1) as the Ricci flow equation and $(M, g(t))$ its solution.

Suppose a one-parameter family of metric $g(t) = \lambda(t) g_0$ is a solution of the Ricci flow equation under Einstein metric $R_{ij} = kg_0$, where $k \in \Re$, we have

$$\frac{\partial g}{\partial t} = \lambda'(t) g_0 = -2kg_0 \qquad (2.2)$$

Solving the resulting ODE, $\lambda'(t) = -2k, \lambda(0) = 1,$ we obtain $g(t) = (1 - 2kt) g_0$ as a solution which exists only for the short time $t \in \left[0, \frac{1}{2}k\right)$ with a singularity at $t = \frac{1}{2}k$. If we consider for instance a unit round sphere $\mathbb{S}^n$ with the standard metric $g_{\mathbb{S}^n}$, then $g(t) = (1 - 2(n-1)) g_{\mathbb{S}^n}$ is a solution to the Ricci flow, implying that the sphere shrinks homothetically under the Ricci flow and eventually collapses to a point in a finite time $T = \frac{1}{2(n-1)}$. Similarly, in the case of hyperbolic manifold with metric g_0 where $Rc(g_0) = -(n-1) g_0$, the evolution $g(t) = (1 + 2(n-1)t) g_0$ will expand the manifold for all time and goes back to time $T = -\frac{1}{2(n-1)}$ upon which the metric explodes out of a single point. While the Ricci flat manifold is steady under the Ricci flow.

The Ricci flow equation (2.1) does not actually preserve the volume whereas in application, we usually prefer the volume to remain fixed to avoid the problem of manifold shrinking or expanding at time t goes to singular time. We however have normalized Ricci flow (2.3) which helps to achieve this

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2r}{n} g_{ij} \qquad (2.3)$$

Here $r = (Vol_g)^{-1} \int_{M^n} R d\mu$ is a constant, the average of the scalar curvature R of M^n, and $Vol_g = \int_{M^n} d\mu$. The factor r appearing in (2.3) keeps the volume of the manifold constant.

On surfaces M^2, the normalized Ricci flow becomes

$$\frac{\partial}{\partial t} g = (r - R) g \qquad (2.4)$$

where $r = A^{-1} \int_{M^2} R dA$, the average of the scalar curvature R of M^2, A is the total surface area, dA is the area element of metric g on M^2 and

$$\frac{\partial}{\partial t} A = \int_{M^2} (r - R) dA = 0 \qquad (2.5)$$

Thus, the total surface area is preserved along the flow. The integral of R over the surface M^2 gives the Euler class $\chi(M^2)$. Recall the Gauss-Bonnet formula (2.6) on closed surface (M^2)

$$\frac{1}{2\pi} \int_{M^2} K dA = \chi(M^2) = 2(1 - \gamma(M^2)) \qquad (2.6)$$

where $\chi(M^2)$ is the Euler characteristic, $\gamma(M^2)$, the genus and K the Gaussian curvature of M^2. Here $2K = R$, then

$$\int_{M^2} R dA = 4\pi\chi(M^2) \qquad (2.7)$$

In fact, Gauss Bonnet formula accounts for the relation between the topology and geometry of the underlying manifold as we can see that the sign of r can be determined explicitly even independent of g,

$$r = \chi(M^2) = \frac{1}{4\pi} \int_{M^2} R dA \qquad (2.8)$$

For example, if we consider a topological 2-sphere whose genus is 0, then $\chi(M^2) = 2$ and $\int_{S^2} R d\mu = 8\pi$.

3.0 Uniformization of Surface

Uniformization theorem implies that every smooth surface admits a unique conformal metric, which classifies surface into three families using the sign of the curvature. This is a classical result in Riemannian geometry, that is, every simply connected surface is conformally equivalent to one of the Riemann sphere, the complex plane and the open disk. In this direction, Ricci flow helps greatly in the classification of closed two-dimensional manifolds into three families of constant positive, zero or negative curvature, as it is also used in the classification of closed three manifold (Geometrization Conjecture) which consequently leads to the proof of Poincaré conjecture. The procedure is to run Ricci flow on smooth surface and allow the metric to be deformed over time such that the scalar curvature evolves as reaction-diffusion equation and eventually becomes constant. The limiting metric is the uniformizing metric which classifies the universal covering space of the surface into one of the three canonical geometries. We now state an important result of Ricci flow on surfaces due to Hamilton [6].

Theorem 3.1. *Let* (M^2, g_0) *be a closed surface, there exists a unique solution* $g(t)$ *of (2.4) for all time* t. *Moreover, at* $t \to \infty$, *the metric* $g(t)$ *converges to a metric* g_∞ *of constant curvature when* $r \le 0$. *If* $R(g_0) > 0$, *then the metric* $g(t)$ *converges to a positive constant curvature metric at time* $t \to \infty$.

The above result together with the work of Chow [7] give a complete proof of the uniformization of surfaces using the Ricci flow. The main point of contention here lies in the class of positive Euler characteristics where the existence of gradient shrinking soliton (see next section) uses Kazdan-Warner identity which itself assumes the Uniformization theorem. (A new proof of Uniformization theorem without Kazdan-Warner identity is in [8]). We remark that Kazdan-Warner identities have played significant roles in understanding blow up behaviour of solution to geometric PDEs that prescribed the curvatures in a conformal class. They were originally introduced by Kazdan and Warner [16] in their study of prescribing Gauss curvature for n-sphere as follows:

$$\int_{S^n} \langle \nabla x_j, \nabla K(x) \rangle dV_g = 0, \quad \text{for} \quad j = 1, \ldots, n+1 \tag{3.1}$$

where x_j are the coordinate function of $S^n \to R^{n+1}$, $K(x)$ is the prescribed function to the scalar curvature of conformal metric g. This identity has been extended to compact manifolds involving Killing vector field X by Bourguion and Ezin [17]

$$\int_{S^n} \langle X, \nabla R_g \rangle dV_g = 0 \tag{3.2}$$

4.0 Existence of Unique Surface Solitons

The Ricci solitons are special self-similar solutions of the Ricci flow and they may be regarded as fixed points of the normalized Ricci flow. They provide motivation to consider certain quantities that may guide us in developing estimates for general solutions. The readers can see for instance [18], [2] and [3] for more information. We call a solution $g(t)$ of the normalized Ricci flow on a surface M^2 a Ricci soliton, if the metric $g(t)$ changes by the pull back diffeomorphism for any times t_1 and t_2 in the interval of existence and any time-dependent scale factor $\sigma(t)$

$$g(t_2) = \sigma(t)\varphi(t)^* g(t_1) \tag{4.1}$$

Starting at time $t = 0$, we have for any time t

$$g(t) = \sigma(t)\varphi(t)^* g_0 \tag{4.2}$$

This simply means that in a Ricci soliton, all the Riemannian manifolds (M^n, g) are isometric up to a scale factor that is allowed to vary with time. Combining (4.2) with scale and diffeomorphism invariance of the flow, we have

$$\frac{\partial g}{\partial t} = L_X g \tag{4.3}$$

where X is the one parameter family of vector field generated by $\varphi(t)$. We then have formally the Ricci Soliton equation

$$2Rc(g_0) + L_X g + \sigma g_0 \tag{4.4}$$

and we have

Definition 4.1. *Suppose* $X = \nabla f$, *a gradient of some smooth function* $f(x,t)$, *we say* $g(t)$ *is a gradient Ricci soliton and satisfies*

$$Rc(g_0) + \nabla\nabla f + \frac{\sigma}{2} g_0 \tag{4.5}$$

We say that g_0 *is shrinking , steady or expanding on depending whether* $\sigma < 0$, $\sigma = 0$

or $\sigma > 0$

Theorem 4.2. *([3] Theorem 4.1). If* g_0 *is a complete gradient Ricci soliton, then there exists a solution* $g(t)$ *of the Ricci flow with* $g(0) = g_0$*, one parameter family of diffeomorphisms* $\varphi(t)$ *generated by* $X = \frac{1}{\tau(t)} grad_{g0} f_0$ *with* $\varphi(0) = I\, d_M$*, function* $f(t)$ *with* $f(0) = f_0$ *defined for all time* t *with*

$$\tau(t) := \in t + 1 > 0$$

such that

$$g(t) = \tau(t)\, \varphi(t)^* g_0$$
$$f(t) = \varphi(t)^* f_0 = f_0 \circ \varphi(t)$$

then also

$$Rc\,(g(t)) + \nabla^{g(t)} \nabla^{g(t)} f + \frac{\in}{2\tau} g(t) = 0 \qquad (4.6)$$

and

$$\frac{\partial f(t)}{\partial t} = \left| grad_{\,g(t)} f(t) \right|^2_{g(t)} \qquad (4.7)$$

Therefore

Definition 4.3. *A solution* $(M^n, g(t)), t \in (-\infty, 0]$ *is said to be a shrinking gradient Ricci soliton if*

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} = 0 \qquad (4.8)$$

Note that equation (2.1) is equivalent to

$$\frac{\partial g_{ij}}{\partial t} = \left(L_{\nabla fg} \right)_{ij} - \frac{1}{|t|} g_{ij}$$

Thus by equation (4.6) with $g(-1) = g$, $g(t) = -t\varphi(t)^* g(-1)$ and $f(t) = f(-1) \circ \varphi(t)$, $g(t)$ is a gradient shrinking soliton which shrinks homothethically up to isometry. Einstein metric of positive scalar curvature provides a very good example. In fact, we know [18], that any gradient shrinking Ricci soliton in dimensions 2 and 3 are Einstein solutions with positive scalar curvature. Gaussian metric (R^2, g_0) is a gradient shrinking soliton with potential $f = \frac{|x|^2}{4}$ *and* $\nabla \nabla f = \frac{1}{2} g_0$. Round cylinder $S^{n-k} \times R^k$ or its quotient provides another class of example of gradient shrinking solitons.

Let's briefly consider some consequences of gradient Ricci soliton equation. Taking the trace of equation (4.6), we have

$$R + \Delta f + \frac{n \in}{2} = 0 \qquad (4.9)$$

adding this with equation (4.7) implies

$$\frac{\partial f}{\partial t} = -\Delta f - R + |\nabla f|^2 - \frac{n \in}{2} \qquad (4.10)$$

Taking the divergence of (4.6), using contracted second Bianchi identity (7.3) and the Ricci identity (7.4), we have

$$0 = g^{jk} \nabla_k \left(R_{ij} + \nabla_i \nabla_j f + \frac{\in}{2} g_{ij} \right) = \frac{1}{2} \nabla_i R + \nabla_i \Delta f - R_{jijk} \nabla_k f \qquad (4.11)$$
$$= -\frac{1}{2} \nabla_i R + R_{ik} \nabla_k f$$

which implies that

$$R + |\nabla f|^2 = \in f \equiv Const \qquad (4.12)$$

constant in the space variables. This can be seen by substituting (4.6) into (4.11), that is,

$$\nabla_i \left(R + |\nabla f|^2 + \in f \right) = 0$$

consequently by (4.9) we have

$$R + 2\Delta f - |\nabla f|^2 - \in f \equiv Const.$$

Now, on the surface M^2 a gradient Ricci soliton is equivalent to

$$(L_X g)_{ij} = (r - R)g_{ij} \tag{4.13}$$

Which implies

$$\Delta f = r - R \tag{4.14}$$

Hence the smooth function f is called the potential of the curvature. On a compact surface, the potential is unique up to addition of constant of time alone. For example when M^2 is diffeomorphic to S^2, the conformal group of S^2 gives the gradient soliton on S^2.

Note that the Kazdan-Warner Identity itself assumes Uniformization theorem. The proof of the above theorem without Kazdan-Warner identity is presented in [2, 8]. We remark that conformal killing vector field is conformally invariant, then in each conformal class, there exists a metric of constant curvature on M^2. Recall that $r \equiv \chi(M)$ by the Gauss-Bonnet formula. It turns out that whenever $r \le 0,$ there are no conformal killing vector fields, though all the Ricci solitons have constant curvatures. We remark also that Kazdan-Warner Identities have been extended to higher dimension [17]. This also implies no conformal Killing vector fields for nonpositive Ricci curvature for $n \ge 0$ [2]. Indeed, either S^2 with usual round metric or its Z_2 quotient RP^2 is the only shrinking gradient Ricci soliton in dimension 2. Here we know that $\chi(M^2) > 0$ and we can assume that M^2 is diffeomorphic to S^2 by passing to the universal cover. We employ Kazdan-Warner Identity (see appendix or [16] for details) to establish this.

Proposition 4.4. *([2], [3]) Let $(M^2, g(t))$ be a shrinking gradient soliton on a closed surface, then $g(t)$ is a metric of constant positive curvature.*

Proof. By a gradient Ricci soliton equation (4.13), we have

$$2\, div\ X = 2(r - R)$$

Since X is a conformal Killing vector field, we obtain

$$\int_{M^2} R(r - R)\ dA = \int_{M^2} (R - r)^2\ dA = -\int_{M^2} R\ divX\ dA$$

Integrating by parts and using the Kazdan-Warner identity (7.6), we obtain

$$\int_{M^2} (R - r)^2 dA = 0$$

Denote the trace free part of the Hessian of the potential function f of the curvature by

$$M_{ij} = \nabla_i \nabla_j f - \frac{1}{2}\Delta f g_{ij} \tag{4.15}$$

It turns out that any metric g_{ij} with $M_{ij} = 0$ is a Ricci soliton. It is now clear that on closed surface of positive curvature, there are no other soliton than the one with constant curvature.

However, there do exist soliton metrics with positive scalar curvature on noncompact surface. For example, the Hamilton cigar soliton where

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

and with potential function $f = -\log(1 + x^2 + y^2)$ has positive Gauss curvature which flows by conformal dilation and is asymptotic to a flat cylinder of finite circumference at infinity.

5.0 Monotonicity of Hamilton's Entropy and its Consequences

Suppose we have a solution of the normalized Ricci flow on a closed surface M^2 with $R > 0$ for all $t > 0$, we can denote a quantity

$$Q = \frac{\partial L}{\partial t} - |\nabla L|^2 = \Delta L + R - r \tag{5.1}$$

where $L = \log R$

Then

$$Q = \frac{\Delta R}{R} - \frac{|\nabla R|^2}{R^2} + (R - r)$$

In [6] Hamilton defines a quantity

$$Z = r^{-1} \int_{M^2} QRdA$$

by definition of Q

$$Z = r^{-1} \int_{M^2} \frac{|\nabla R|^2}{R} + R(R - r)$$

$$\frac{dZ}{dt} = r^{-1} \int_{M^2} \left(-\frac{2(\nabla R, \nabla R_t) + (r - R)|\nabla R|^2}{R} + R_t (R - r) + RR_t \right) dA$$

$$+ \int_{M^2} \frac{|\nabla R|^2}{R} + R(R - r)(r - R)\, dA$$

Using $R_t = \Delta R + R\,(R - r)$, we have

$$\frac{\partial Z}{\partial t} \geq Z^2 + rZ \qquad (5.2)$$

Suppose Z is positive at a particular time t_0, then it would blow up in a finite time. This contradicts the fact that solution exists for all time, then we have that $Z \geq 0$. However, we have the following Hamilton entropy

$$H\,(g) := \int_{M^2} R \log R \, d\mu \qquad (5.3)$$

defined for a metric of strictly positive curvature on a compact surface. We hereby follow the argument in [6] that this entropy is decreasing under the normalized Ricci flow. Chow [7] has modified this quantity to extend to the case where the curvature of g_0 changes sign.

Now computing the time derivative of (5.3), it then follows by direct calculation

$$\frac{dH(g)}{dt} = \frac{\partial}{\partial t}\left(\int_{M^2} R \log R \, dA\right)$$

$$= \int_{M^2} \left(\frac{\partial}{\partial t} R\right) \log R \, dA + \int_{M^2} R \left(\frac{\partial}{\partial t} \log R\right) dA + \int_{M^2} R \log R \frac{\partial}{\partial t} dA$$

$$= \int_{M^2} [\Delta R + R(R - r)] \log R \, dA + \int_{M^2} \frac{\partial}{\partial t} R \, dA + \int_{M^2} R \log R (r - R)\, dA$$

$$= \int_{M^2} [\Delta R + R(R - r)] \log R \, dA + \int_{M^2} [\Delta R + R(R - r)]\, dA - \int_{M^2} R\,(R - r) \log R \, dA$$

$$= \int_{M^2} [\Delta R + R(R - r)]\, dA + \int_{M^2} \Delta R \log R \, dA$$

$$= \int_{M^2} R\,(R - r)\, dA - \int_{M^2} \frac{|\nabla R|^2}{R} dA$$

The last inequality was obtained by integration by parts methods

Proposition 5.1. *If* $(M^2, g(t))$ *is a normalized Ricci flow on compact surface such that* $R(g_0) > 0$, *the Hamilton entropy satisfies*

$$\frac{dH\,(g)}{dt} = -\int_{M^2} \frac{|\nabla R + R\nabla f|^2}{R} dA - 2\int_{M^2} |M|^2 \, dA \qquad (5.4)$$

where M *is the trace-free part of the Hessian of potential function* f. *Moreover, the entropy is strictly decreasing unless* $g(t)$ *is a gradient Ricci soliton.*

Proof. Recall that $M = \nabla\nabla f - \frac{1}{2}\Delta f \; g$ *and* $\Delta f = R - r$

$$\int_{M^2} |M|^2 \, dA = \int_{M^2} \left(|\nabla\nabla f|^2 - \frac{1}{2}(\Delta f)^2 \right) dA$$

and

$$-2\int_{M^2} |M|^2 \, dA = \int_{M^2} \left((\Delta f)^2 - 2|\nabla\nabla f|^2 \right) dA \tag{5.5}$$

Using integration by parts and the Ricci identity

$$\begin{aligned}
\int_{M^2} (R-r)^2 \, dA = \int_{M^2} (\Delta f)^2 &= -\int_{M^2} \langle \nabla f, \nabla \Delta f \rangle \, dA \\
&= -\int_{M^2} \langle \nabla f, \nabla \Delta f - Ric(\nabla f) \rangle \, dA \\
&= -\int_{M^2} \left(\langle \nabla f, \nabla \Delta f \rangle - Ric(\nabla f, \nabla f) \right) dA \\
&= -\int_{M^2} \left(\langle \nabla\nabla f, \nabla\nabla f \rangle + \frac{1}{2} R(\nabla f, \nabla f) \right) dA \\
&= -\int_{M^2} \left(|\nabla\nabla f|^2 + \frac{1}{2} R |\nabla f|^2 \right) dA
\end{aligned}$$

Putting this back into (5.5), we have

$$\begin{aligned}
-2\int_{M^2} |M|^2 \, dA &= \int_{M^2} \left(\frac{1}{2} R|\nabla f|^2 - |\nabla\nabla f|^2 \right) dA \\
&= \int_{M^2} \left(R|\nabla f|^2 - (R-r)^2 \right) dA
\end{aligned} \tag{5.6}$$

(since $|\nabla\nabla f|^2 = (R-r)^2 - \frac{1}{2} R|\nabla f|^2$). Expanding and integrating by parts we have

$$\begin{aligned}
-\int_{M^2} \frac{|\nabla R + R\nabla f|^2}{R} dA &= -\int_{M^2} \left(\frac{|\nabla R|^2}{R} + 2\nabla R \nabla f + R|\nabla f|^2 \right) dA \\
&= -\int_{M^2} \left(\frac{|\nabla R|^2}{R} + 2R\Delta f + R|\nabla f|^2 \right) dA \\
&= -\int_{M^2} \left(\frac{|\nabla R|^2}{R} + 2R(R-r) + R|\nabla f|^2 \right) dA
\end{aligned} \tag{5.7}$$

Adding (5.6) and (5.7), we obtain

$$-2\int_{M^2} |M|^2 \, dA - \int_{M^2} \frac{|\nabla R + R\nabla f|^2}{R} dA = \int_{M^2} \left((R-r)^2 - \frac{|\nabla R|^2}{R} \right) dA \tag{5.8}$$

as the desired result. We also note that equality in (5.8) holds if $M \equiv 0$, thus the flow is a gradient Ricci Soliton. Note that if equality is attained in (5.4), then $M \equiv 0$ which implies that the solution is a gradient Ricci soliton and $R \equiv r$ is constant.

Corollary 5.2. *With the conditions of Proposition (5.1), the Hamilton entropy is a strictly decreasing function of time unless* $R(\cdot, 0) \equiv r$ *, in which case, it is constant in time.*

Proof. If

$$\frac{dH}{dt} = 0$$

at some time $t_0 \in [0, \infty)$, which implies that

$$M(\cdot, t_0) \equiv 0$$

then $g(t_0)$ is a gradient Ricci soliton. Hence $R(\cdot, t_0) \equiv r$ is constant and thus $g(t) \equiv g(t_0)$. In general, we have a uniform constant $C > 0$ such that the Hamilton entropy

$$\int_{M^2} R \log R \, dA \le C \tag{5.9}$$

Finally, we have

Theorem 5.3. *With the conditions of Proposition (5.1), the Hamilton entropy is a strictly decreasing unless the solution* $g(t)$ *is a gradient shrinking soliton.*

Proof. From equation (4.15)

$$M_{ij} = \nabla_i \nabla_j f - \frac{1}{2}\Delta f g_{ij} = \nabla_i \nabla_j f + R_{ij} - \frac{1}{2\tau} g_{ij}$$

where we have used $\Delta f = r - R \;\; and \;\; r = \frac{1}{\tau}$. Also by equation (4.11)

$$0 \equiv \left|\nabla R + R\nabla f\right|^2 = div\left(Rc + \nabla\nabla f - \frac{1}{2\tau} g\right),$$

we can then write (5.4) as

$$\frac{dH}{dt} = -2\int_{M^2}\left|\nabla_i \nabla_j f + R_{ij} - \frac{1}{2\tau} g_{ij}\right|^2 - \int_{M^2} \frac{\left|div\left(Rc + \nabla\nabla f - \frac{1}{2\tau} g\right)\right|^2}{R} \tag{5.10}$$

6.0 LYH Differential Harnack Inequalities on Surfaces

In this section, we make brief remarks on Harnack inequality of Li-Yau-Hamilton type as a by-product of the last section. We restrict the discussion to a surface of positive scalar curvature, though, the result works in general for the case of nonnegative scalar curvature. Note that differential Harnack inequalities for evolution equations began with a celebrated paper of Li and Yau [11] for heat operator on Riemannian manifolds. Their approach is based on the Maximum principle, and since this works, it has been proven for many geometric evolution partial differential equations using similar approach. Most notable works in the direction of this present note are Hamilton's [9], Chow and Hamilton's [14] and Perelman's [10] for conjugate heat equation. Cao [12], Chow [13], and Hamilton [15] have also extended Li-Yau-Harnack inequalities to the cases of Kahler-Ricci flow, Gaussian curvature flow and mean curvature flow respectively.

In the present, we follow Perelman's idea of coupled Ricci flow-conjugate heat equations

$$\begin{cases} \partial_t u = -\Delta u + Ru \\ \partial_t g_{ij} = -2R_{ij} \end{cases} \tag{6.1}$$

where $u = u(t, x) \in (0, T) \times M, T > 0$. His monotonicity formula implies, pointwise gradient estimates for the fundamental solution of the conjugate heat equation and f is a function such that $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f} \;\; with \;\; \tau = T - t$ backward time, namely

$$P = \left[\tau\left(2\Delta f - |\nabla f|^2 + R\right) + f - n\right]u \le 0$$

Satisfies

$$(\partial_t + \Delta - R)P = -2\tau\left|\nabla_i \nabla_j f + R_{ij} - \frac{1}{2\tau} g_{ij}\right|^2 \le 0$$

Let $g(t)$ be a solution to the Ricci flow on surface M^2 with the scalar curvature $R > 0$

$$\partial_t g_{ij} = -Rg_{ij} \tag{6.2}$$

The scalar curvature then evolves by

$$\partial_t R = \Delta R + R^2 \tag{6.3}$$

Hamilton [9] proves that if the initial metric has positive scalar curvature, then

$$\partial_t In\ R - \left|\nabla\ In\ R\right|^2 + \frac{1}{t} = \Delta\ In\ R + R + \frac{1}{t} > 0$$

As a by-product of the monotonicity discussed in the last section, we obtain the following

Theorem 6.1. *Let* $(M, g(t))$ *be a solution to the Ricci flow (6.2) on a surface with positive scalar curvature* $R > 0$, *let U be a positive solution to the conjugate heat equation*

$$\partial_t u - \Delta u + Ru = 0$$

such that $f = -\ln u$, $\tau = T - t$, *then* f *satisfies*

$$\partial_t f = -\Delta f + |\nabla f|^2 - R$$

and

$$\frac{|\nabla u|^2}{u^2} - \frac{2u_t}{u} - R \le \frac{1}{T-t}$$

Proof. We give the sketch of the proof

By standard smoothness argument $f = -In\, u$ implies $\partial_t f = -\Delta f + |\nabla f|^2 - R$. By Theorem (5.3), if H is nonincreasing, then $\left|\nabla_i \nabla_j f + R_{ij} - \frac{1}{2(T-t)} g_{ij}\right|^2 \ge 0$, which may imply $\nabla_i \nabla_j f + R_{ij} - \frac{1}{2(T-t)} g_{ij} \le 0$ or $\nabla_i \nabla_j f + R_{ij} - \frac{1}{2(T-t)} g_{ij} \ge 0$, but we know that the later holds by the Pereman's $W-entropy$ monotonicity and it is strictly positive in all dimensions, except when $g(t)$ is a shrinking gradient soliton. Notice also that by Cauchy-Schwarz inequality

$$\left|\Delta f + R - \frac{n}{2}\right|^2 = \left|g^{ij}\left(\nabla_i \nabla_j f + R_{ij} - \frac{1}{2\tau} g_{ij}\right)\right|^2 \le n\left|\nabla_i \nabla_j f + R_{ij} - \frac{1}{2\tau} g_{ij}\right|^2$$

Since $\sum_{ij} g_{ij}^2 = n$

Here $n = 2$, using the nonnegativity of Einstein tensor (See Appendix A.2 for detail), we have

$$\Delta f + R \le \frac{1}{(T-t)} \qquad (6.4)$$

We recall on the other hand that $\int_{M^2} \left(\Delta f - |\nabla f|^2\right) e^{-f} = 0$, since M^2 is closed, which implies

$$\Delta f - |\nabla f|^2 = 0 \qquad (6.5)$$

Adding (6.4) and (6.5) gives

$$2\Delta f - |\nabla f|^2 + R \le \frac{1}{T-t}$$

Direct computations gives

$$|\nabla f|^2 = \frac{|\nabla u|^2}{u^2} \quad and \quad \Delta f = \Delta(-In\ u) = \frac{|\nabla u|^2}{u^2} - \frac{\Delta u}{u} = \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - R$$

Hence

$$2\Delta f - |\nabla f|^2 + R = \frac{|\nabla u|^2}{u^2} - \frac{2u_t}{u} - R$$

And the estimate follows immediately

Corollary 6.2. *With the same conditions as in the above Theorem 6.1, we have for any two points* (t_1, x_1) *and* (t_2, x_2) *in* $(0, T) \times M^2$ *such that* $t_1 < t_2$, *that*

$$u(t_1, x_1) \le u(t_2, x_2)\left(\frac{\tau_1}{\tau_2}\right)^2 \exp\left[\frac{\int_{t_1}^{t_2}\left(\frac{1}{4}|\gamma(s)|^2 + (\tau_2 - \tau_1) R\right.}{2(\tau_2 - \tau_1)}\right]$$

Here is the time derivative, $\tau_i = T - t_i$, $i = 1, 2$ *and* $\gamma(s) : [0, 1] \to M^2$ *is a smooth curve joining the points* (t_1, x_1) *and* (t_2, x_2).

Proof. Let $x_1, x_2 \in M^2$ *and* $\gamma : [0, 1]$ be a smooth curve joining x_1 *and* x_2. Suppose we define $L(s) = In\,(\gamma(s), T - \tau(s))$ such that $\tau(s) = (1-s)\tau_1 + s\tau_2,\ 0 \le \tau_2 < \tau_1 \le T$ and $L(0) = In\, u\,(x_1, \tau_1)$ *and* $L(1) = \ln u(x_2, \tau_2)$ Then

$$\frac{\partial L(s)}{\partial s} = \frac{\partial_s U}{u} = \frac{\nabla u}{u}\dot{\gamma}(s) - \frac{\partial_\tau u(\tau_1 - \tau_2)}{u}$$
$$= (\tau_1 - \tau_2)\left(\frac{\nabla u}{u}\cdot\frac{\dot{\gamma}(s)}{(\tau_1 - \tau_2)} - \frac{1}{u}\frac{\partial u}{\partial \tau}\right)$$
$$\leq (\tau_1 - \tau_2)\left(\frac{2\dot{\gamma}(s)}{(\tau_1 - \tau_2)} + \frac{|\nabla u|^2}{2u^2} - \frac{1}{u}\frac{\partial u}{\partial \tau}\right)$$
$$= \frac{2|\dot{\gamma}(s)|^2}{(\tau_1 - \tau_2)} + \frac{\tau_1 - \tau_2)}{2}\left[\frac{|\nabla u|^2}{u^2} - \frac{2}{u}\frac{\partial u}{\partial \tau}\right]$$

By the last Theorem 6.1, we have that

$$\frac{|\nabla u|^2}{u^2} - 2\frac{u_r}{u} - R \leq \frac{1}{T-t}$$

with $R \geq 0$. Then

$$\frac{\partial L(s)}{\partial s} \leq \frac{2|\dot{\gamma}(s)|^2}{(\tau_1 - \tau_2)} + \frac{(\tau_1 - \tau_2)}{2}\left(R + \frac{1}{T-t}\right)$$

By the fundamental theorem of calculus

$$In\ \frac{u(x_2, t_2)}{u(x_1, t_1)} = In\ \ u(\gamma(s), s)\Big|_0^1 = \int_0^1 \frac{\partial L(s)}{\partial s}ds$$
$$\leq \frac{2\int_0^1 |\dot{\gamma}(s)|^2 ds}{(\tau_1 - \tau_2)} + \frac{(\tau_1 - \tau_2)}{2}\int_0^1 Rds + \frac{1}{2}In\ \frac{T - t_1}{T - t_2}$$

Hence, for any two points (x_1, t_1) *and* (x_2, t_2) in the space-time, we have by exponentiation of the last inequality that

$$u(t_1, x_1) \leq u\ (t_2, x_2)\left(\frac{\tau_1}{\tau_2}\right)^2 \exp\left[\frac{\int_{t_1}^{t_2}(4|\gamma(s)|^2 + (\tau_2 - \tau_1)R}{2(\tau_2 - \tau_1)}\right]$$

7.0 Appendix

A.1. Laplace-Beltrami Operator

Let (M, g) be an n-dimensional manifold equipped with Riemannian metric g. let f be a smooth function on M, then the laplacian acting on f, called Laplace Beltrami operator is defined as divergence of the gradient of f, i.e.

$$\Delta = div\ grad\ f$$

Suppose (M, g) is an oriented smooth manifold with the volume $Vol_g = \mu_g = \sqrt{|g|}dx^i$. The divergence of a vector field X on the manifold is the scalar function

$$(div\ X)\mu_g = L_X \mu_g$$

where $(L)_X$ is the Lie derivative along the vector field X. Then

$$div\ X = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}X^i\right)$$

while the gradient of f is defined as

$$(grad\ f)^i = \partial^i f = g^{ij}\partial_i f$$

Therefore $$\Delta f = div\ grad\ f = \frac{1}{\sqrt{|g|}}\partial_i\left(g^{ij}\sqrt{|g|}\,\partial_j f\right) \qquad (7.1)$$

A.2 Einstein Metric

A Riemannian metric is said to be Einstein if its Ricci tensor is a scalar multiple of the metric at each point, that is, for some constant λ

$$R_{ij}(g) = \lambda g_{ij} \qquad everywhere$$

So Einstein condition is simply written as

$$R_{ij}(g)=\frac{1}{n}Rg_{ij} \tag{7.2}$$

Recall that the Einstein tensor G is a rank 2 tensor field

$$G_{ij} = R_{ij}(g) - \frac{1}{2}Rg_{ij}$$

contracting with inverse metric g^{ij}, we have

$$0 \equiv G = \frac{2-n}{2}R$$

Thus, in dimension $n \neq 2$, this implies that $R \equiv 0$ on M, which is Ricci flat manifold
The Contracted Second Bianchi Identity is given as

$$g^{ij}\nabla_i R_{jk} = \frac{1}{2}\nabla_k R \tag{7.3}$$

which is equivalent to the Einstein tensor being divergence-free

$$div\left(R_{ij}(g)-\frac{1}{2}Rg_{ij}\right)=0$$

Consider the commutators of Δ *and* ∇ on any function f, we have

$$\Delta\nabla_i f = \nabla_j\nabla_i\nabla_j f = \nabla_i\nabla_j\nabla_j f - R_{ijkl}\nabla_k f$$

Which implies that

$$\Delta\nabla_i f = \nabla_i\Delta f + R_{ij}\nabla_j f \tag{7.4}$$

By Bochner identity, we identify

$$\Delta|\nabla f|^2 = 2|\nabla\nabla f|^2 + 2R_{ij}\nabla_i f\nabla_j f + 2\nabla_i f\nabla_i(\Delta f) \tag{7.5}$$

A.3 Killing Vector Field

Let $\varphi(t) : M \to M$ *be a one-parameter family of diffeomorphism generated by a vector field* X. *Define the pull back diffeomorphism* $\varphi_t^* : T_{\varphi(p)}^* M \to T^* M$ *on the tangent bundles as* $\varphi(t)_t^* = (\varphi^{-1}(t))_*$, *where* $(\varphi_t)_* : T_P M \to T_{\varphi(P)} M$ *is the differential of* $\varphi(t)$, *acting on cotagent bundle by*

$$(\varphi_t)_* = (\varphi_t^{-1}) : T_P M \to T_{\varphi(p)}^* M$$

We define the **Lie derivative** of the metric tensor g_{ij} as

$$\left.\frac{\partial}{\partial t}\right|_{t=0}(\varphi_t^* g)_{ij} = (L_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$$

where ∇ denotes covariant differential operator

We say that a vector field is a Killing Vector field if $L_X g = 0$ and in local coordinate we write

$$\nabla_i X_j + \nabla_j X_i = 0$$

So a vector field is said to be a **conformal Killing vector field**, if

$$(L_X g)_{ij} = \frac{2}{n}div(X)g_{ij}$$

This implies that if $\{\varphi_t\}_{t\geq 0}$ is a group of conformal diffeomorphism which generates vectors X, then

$$\left.\frac{\partial}{\partial t}\right|_{t=0}(\varphi_t^* g)_{ij} = \frac{2}{n}div(X)g_{ij}$$

Theorem 7.1. *If* (M^n, g) *is a closed Riemannian manifold with non positive Ricci curvature, then there are no nontrivial killing vector field.*

Proof. From

$$\left.\frac{\partial}{\partial t}\right|_{t=0}(\varphi_t^* g)_{ij} = (L_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$$

Since φ_t is conformal we have

$$\nabla_i X_j + \nabla_j X_i = \frac{2}{n} div(X) g_{ij}$$

Taking the divergence, we have

$$\Delta_i X_j + R_{jk} X_k + \nabla_j \; div \; X = \frac{2}{n} div(X) g_{ij}$$

$$\Delta_i X_j + R_{jk} X_k + \left(1 - \frac{2}{n}\right) \nabla_j \; div \; X = 0$$

Hence

$$\frac{1}{2} \Delta |X| = |\nabla X|^2 + \langle \Delta X, X \rangle$$

$$= |\nabla X|^2 - Rc(X, X) - \left(1 - \frac{2}{n}\right) \langle \nabla \; div \; X, X \rangle$$

Integrating by parts, we have

$$0 = \int_M \left(|\nabla X|^2 - Rc(X, X) + \left(1 - \frac{2}{n}\right) |div \; X|^2 \right) d\mu$$

Since each of the three terms on RHS are nonnegative and $Ric < 0$, we conclude that $X \equiv 0$.

Theorem 7.2. *(Kazdan-Warner Identity [16, 17]) If* X *is a conformal Killing vector field on* S^2*, then*

$$\int_{M^2} \langle \nabla R, X \rangle \, dA = \int_{M^2} R \; div \; X \; dA = 0 \qquad (7.6)$$

Here $\chi(M^2) > 0$*, so by passing to the universal cover we may assume that* M^2 *is diffeomorphism to* S^2.

8.0 Conclusion

We have discussed how two dimensional compact Riemannian manifold can be deformed using conformal Ricci flow, a nonlinear geometric evolutionary Partial Differential Equation, the consequence of which is a complete proof of a classical theorem in Topology, the uniformization theorem of Poincaré and Koebe. We then focused attention on the surfaces of genus zero where we made some important remarks. It was established that either the round 2-sphere, S^2 or its Z_2 quotient, RP^2 is the only gradient shrinking Ricci Soliton via Hamilton surface entropy monotonicity. As a by product, we derived Li-Yau-Hamilton type (LYH) differential Harnack estimates on positive solutions of conjugate heat equation defined on a surface with nonnegative scalar curvature.

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