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Higher Order Nonsingular Immersions of Dold Manifolds.

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Abstract

In this paper we employ β – operations and characteristic classes to study nonexistence of higher order nonsingular immersions of Dold manifolds into a Euclidean space.

1.0 Introduction

In [1] ,[2] and [3], Atiyah, Feldman and Pohl have considered higher order tangent bundles of a smooth manifold M and the higher order nonsingular immersion of M into Euclidean spaces. In [4], [5] and [6], Suzuki obtained some higher order non-immersion theorems of projective spaces into Euclidean spaces or projective spaces by means of characteristic classes, β – operations and spin operations. In[7] ,[8] and[9], Khare, Mukerjee and Yoshioka obtained complete formulas of Stiefel-Whitney classes of higher order tangent bundles of complex projective spaces and Dold manifolds and applied the results to higher order non-immersions of these spaces. The purpose of this paper is to prove a higher order non-immersion theorem for Dold manifolds, using β-operations and characteristic classes.

Preliminaries.

Let M be an n-dimensional smooth manifold. Suppose $T_q(M)$ be the q^{th} order tangent bundle of M , then $T_q(M)$ is a smooth V(n,q)–vector bundle, where v(n,q) $=\binom{n+q}{q}$-1 . Set $T_q^0(M)$ =V(n,q) $-T_q(M)$ in [KO]~(M).Let λ^i ,β^i and ℵ-dim be as defined in [1]. Let $W^q(M)$, $\hat{W}^q(M)$, $Wi^q(M)$, and $\hat{w}i^q(M)$ be total, dual total, i – dimensional and dual i-dimensional Stiefel – Whitney class of $T_q(M)$ respectively. Let $\subseteq_q$ denoteqth order nonsingular immersion and $\not\subseteq_q$ its negative. We have the following theorem.

Theorem 1.Following Mukerjee [8] and Suzuki [4].

a.If $M\subseteq R^{v(n,q)+u}$, then $\hat{W}^q_i= 0$ for $i> u \geqq 0$;

b .If $M \subseteq_q R^{v(n,q)+u}$,then $W^q_i=0$ for $0 \geqq u > - i$;

c. If $M\subseteq_q R^{v(n,q)+u}$, then $\beta^i(T^0_q(M))=0$ for $i>u \geqq 0$;

d. If $M \subseteq_q R^{v(n,q)+u}$,then $\beta^i (- T^0_q(M)) = 0$ for $0 \geqq u > - i$.

Let μ^i : K0(M) → KO(M) { or μ^i : K(m) → K(M) } (i =1,2,3,…) be the symmetric ith power operation which has the following properties from [3].

(i). $\mu^0x =0$, (ii). $\mu^1x = x$,

(iii$\mu i(x + y) = \sum_j^i\mu^j x . \mu^{i-j}y$ for $x,y\epsilon KO(M)$. Then,

Theorem 2. In ([3]), $T_q(M) = \mu^q(T(M) +1) - 1$.

Nonimmersion theorem and its proof.

Let $\mathcal{R}q^m$, $\mathsf{C}q^n$ and $\mathcal{D}$(m,n) be m-dimensional real, n-dimensional complex projective spaces and a Dold maniford

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manifold of type (m,n) respectively. In this section we will show the higher order nonimmersion theorem for $\mathcal{D}$(m,n). Let Υ : K(m) →K0(m) be the realification. Let φ and ά be the canonical line bundles over $\mathcal{R}q^m$ and $\mathbb{C}q^n$, respectively. Let φ and α be the bundles over $\mathcal{D}$(m,n) which are defined in [10]. We have the following:

Proposition 1.In [10] we have a 1- plane bundle φ and a 2 – plane bundle ά over $\mathcal{D}$(m,n) such that

(i). $i^*\varphi = \varphi$, $j * \alpha = \Upsilon(\acute{\alpha})$, $i * \alpha = 1 + \Upsilon$;

(ii). $\varphi \otimes \varphi = 1$, $\varphi \otimes \alpha = \alpha$ where $i : \mathcal{R}q^m \to \mathcal{D}(m,n)$, $j : \mathbb{C}q^n \to \mathcal{D}(m,n)$ are inclusions.

Theorem 3. Using [7] and [10],$\tau(\mathcal{D}(m,n)) \oplus \varphi \oplus 2 = (m+1)\varphi \oplus (n+1)\alpha$. Now, let $i : \mathcal{R}q^m \to \mathcal{D}(m,n)$, $j : \mathbb{C}q^n \to \mathcal{D}(m,n)$ be inclusions. From theorem 2, theorem 3 and the natural property of μ^i, we have,

Theorem 4.

(i) $i^* \tau_q(\mathcal{D}(m,n)) = \sum_{0<odd i \leqq q} \binom{n+q-i-1}{q-i}\binom{m+n+i}{i}\varphi + \sum_{0 \leqq even i \leqq q} \binom{n+q-i-1}{q-i}\binom{m+n+i}{i} - 1$,

(ii) $j * \tau_q(\mathcal{D}(m,n)) = -\sum_{i=0}^{q} \binom{m+q-i-2}{q-i} \mu^i((n+1)\ (\acute{\alpha})) - 1$,

from Theorem 4 we get,

$i * \tau_q^0(\mathcal{D}(m,n)) = -\sum_{0<odd i \leqq q} \binom{n+q-i-1}{q-i}\binom{m+n+i}{i}\Psi$,

where $\Psi = \varphi - 1$. By [1] and [7] ,

we obtain $\beta^i(i * \tau_q^0(\mathcal{D}(m,n))) = \pm 2^{i-1}\binom{G+i-1}{i}\Psi$

and $\beta^i(-i * \tau_q^0(\mathcal{D}(m,n))) = \pm 2^{i-1}\binom{G}{i}\Psi$,

where

$G = \sum_{0<odd i \leqq q} \binom{n+q-i-1}{q-i}\binom{m+n+i}{i}$.

Hence, $\beta^i(i * \tau_q^0(\mathcal{D}(m,n))) = 0 \Leftrightarrow 2^{i-1}\binom{G+i-1}{i} \equiv 0 \bmod 2^{\prod(m)}$

and $\beta^i(-i * \tau_q^0(\mathcal{D}(m,n))) = 0 \Leftrightarrow 2^{i-1}\binom{G}{i} \equiv 0 \bmod 2^{\prod(m)}$,

where $\prod(m)$ = number of integers Ω for $0 < \Omega \leq m$ and Ω =0 ,1 ,2, or 4 mod 8.

Applying the same method as in [9],

we can get $K_i(j * \tau_q(\mathcal{D}(m,n))) = \binom{\delta}{i}\vartheta^i$ and $\check{K}_i(j * \tau_q(\mathcal{D}(m,n))) = \binom{\delta+i-1}{i}\vartheta^i$,

where ϑ = generator of $H^2(\mathbb{C}q^n; \mathbb{Z}_2)$

and $\delta = \frac{1}{2}\sum_{0<odd i \leqq q} \binom{m+q-i-2}{q-i}\binom{2n+i+1}{i}$.

Let us define $g_1 = \max\left\{i \,\middle|\, i > 0, 2^{i-1}\binom{G+i-1}{i} \not\equiv 0 \bmod 2^{\pi(m)}\right\}$,

$g_2 = \max\left\{i \,\middle|\, i > 0,\ 2^{i-1}\binom{G}{i} \not\equiv 0 \bmod 2^{\pi(m)}\right\}$,

$g_1' = \max\left\{i \,\middle|\, 0 < i \leqq m, \binom{G+i-1}{i} \not\equiv 0 \bmod 2\right\}$, $g_2' = \max\left\{i \,\middle|\, 0 < i \leqq m, \binom{G}{i} \not\equiv 0 \bmod 2\right\}$,

$\delta_1 = \max\left\{i \,\middle|\, 0 < i \leqq n, \binom{\delta+i-1}{i} \not\equiv 0 \bmod 2\right\}$,

$\delta_2 = \max\left\{i \,\middle|\, 0 < i \leqq n, \binom{\delta}{i} \not\equiv 0 \bmod 2\right\}$, $f_1 = \max\{g_1, g_1', \delta_1\}$, $f_2 = \max\{g_2 g_2', \delta_2\}$.

Theorem 5. If $-f_2 < B < f_1$, then $\mathcal{D}(m,n) \not\subseteq_q R^{v(m+n,q)+B}$

Proof. Using the natural properties of β^i-operations andStiefel –whitney class, we have that

$\beta^i(i * \tau_q^0(\mathcal{D}(m,n))) \neq 0$, $\beta^i(-i * \tau_q^0(\mathcal{D}(m,n))) \neq 0$,

$K_i(j * \tau_q(\mathcal{D}(m,n))) \neq 0$ and $\breve{K}_i(j * \tau_q(\mathcal{D}(m,n))) \neq 0$

this implies $\beta^i(i * \tau_q^0(\mathcal{D}(m,n))) \neq 0$,

$\beta^i(-i * \tau_q^0(\mathcal{D}(m,n))) \neq 0$,

$K_i^q(\mathcal{D}(m,n))) \neq 0$

and

$\breve{K}_i^q(\mathcal{D}(m,n))) \neq 0$, respectively.

Remarks.

(i) In Theorem 5, if we use Pontrjagin classes instead of Stiefel-whitney classes,then when q =1 we recover the main results of Ucci [10].

(ii) In [6] and [9], Suzuki and Yoshioka obtained the following formula$K^q(\mathcal{D}(m,n)) = (1+\varphi)^{G'}(1+\varphi+\sigma)^{\delta'}$,whereφ, σ are the classes which are defined in [10]. From this formula we obtain the following results.

Theorem 6.

Let $\delta'_1 = \max\left\{i \,\middle|\, 0 < i = \partial + 2\vartheta \leqq m+2n,\ \sum_{0\leqq\beta\leqq min}(\partial,\ \delta'-\vartheta)\binom{G'}{\partial-\beta} * \frac{\delta'!}{(\delta'-\vartheta-\beta)!\beta!\vartheta!} \not\equiv 0 \bmod 2\right\}$.

$\delta'_2 = \max\left\{i \middle| 0 < i = \partial + 2\vartheta \leqq m+2n, \sum_{0\leqq\beta\leqq min}(\partial, 2^u - \delta' - \vartheta)\binom{G'-1+\partial-\beta}{\partial-\beta} * \frac{(2^u-\delta')!}{(2^u-\delta'-\vartheta-\beta)!\beta!\vartheta!} \not\equiv 0 \bmod 2\right\}$.

Where$G' = \frac{1}{2}\sum_{0\leqq eveni\leqq q}\left\{\binom{2n+q+i+1}{i} + (-1)^{\rho-1}\frac{m+2(q-i)-1}{m-1}\binom{n+2^{-1}i}{2^{-1}i}\right\} * \binom{m+q-i-2}{m-2}$,

$\delta' = \frac{1}{2}\sum_{0<oddi\leqq q}\binom{2n+q+i+1}{i}\binom{m+q-i-2}{m-2}$,u is an integer such that $2^u > \max\{mn,\ \delta' - 1\}$.

Ifξ is an integer such that $-\delta'_1 < \xi < \delta'_2$,then$(\mathcal{D}(m,n)) \not\subseteq_q R^{\nu(m+2n,q)\xi}$.Now we give some examples to show that in some cases our theorem 5 can give sharper non immersion results than the above theorem.

(1) When q = 1 , then $G' = m$,

$\delta' = n+1$, $G = m+n+1$, $\delta = n+1$. Let (m ,n) = (14 ,1).

Then $K(\mathcal{D}(m,n)) = (1+\varphi)^{14}(1+\varphi+\eta)^2 = 1$ and theorem 6 gives no information.

By direct calculations we have $f_1 = 4$. So we have :

Corollary1. $\mathcal{D}(14,1) \not\subseteq R^{16+\xi}$, $\xi \leqq 3$. In general , let $(m,n) = (2^y-2^t, 2^t-1)$, $y \geqq t \geqq 0$.

Then $K(\mathcal{D}(m,n)) = (1+\varphi)^{2y-2t}(1+\varphi+\eta)^{2t} = 1$ and theorem 6 gives no information. By direct calculations we have $f_1 \geqq 2^{t-1}(2^{y-t}-1)-y$ if $t \geqq 1$,$f_1 \geqq 2^{y-2}$ if $y \geqq 4$ and t = 0. Then we have:

Corollary 2. (a) If $u < 2^t - 1(2^{y-t}-1) - y$, then $\mathcal{D}(m,n) \not\subseteq R^{m+2m+u}$,where $(m,n) = (2^y-2^t, 2^t-1)$, $y \geqq t \geqq 1$.

(b) In ([1]), if $\xi < 2^{y-2}$, then$\mathcal{D}(m,0) \not\subseteq R^{m+\xi}$,where $m = 2^y-1$, $y \geqq 4$.

(2) when q =2 , then $G' = (n+1)^2 - m$, $\delta' = (n+1)(m-1)$, $G = n(m+n+1)$. Let (m,n) = (12, 3). Then$K^2(\mathcal{D}(m,n)) = (1+\varphi)^4(1+\varphi+\eta)^{44} = 1$ and theorem 6 gives no information. By direct calculations we have $f_1 = 4$, $f_2 ==4$. Thus we obtain:

Corollary 3. $\mathcal{D}(12,\ 3) \not\subseteq_2 R^{189+\xi}$, $-3 \leqq \xi \leqq 3$.

Now, let $(m,n) = (2^y-2^t, 2^t-1)$, $y \geqq t \geqq 1$. Then $K^2(\mathcal{D}(m,n)) = (1+\varphi)^{2t-2y+2t}(1+\varphi+\eta)^{2t(2y-2t-1} = (1+\varphi)^{2y(2t-1)} = 1$ and theorem 6 gives no information. By direct calculations we have $f_1 \geqq 2^{t-1}(2^{y-t}-1)-y$, $f_2 \geqq 2^{t-1}(2^{y-t}-1)-y$. thus we obtain:

Corollary 4. If $-2^{t-1}(2^{y-t}-1)+y < \xi < 2^{t-1}(2^{y-t}-1)-y$, then$\mathcal{D}(m,n)) \not\subseteq_2 R^{\nu(m+2n,\ 2)\xi}$, where $(m,n) = (2^y-2^t, 2^t-1)$, $y \geqq t \geqq 1$.

Conclusion

We used β – operations (Pontrjagin) and Stiefe-Whitney classes to show higher order nonsingular immersions of Dold manifolds. Tangent bundles of manifolds at point q are applied for the prove. $\beta^i(i * \tau_q^0 (\mathcal{D}(m,n))) \neq 0$ and $\beta^i(-i * \tau_q^0 (\mathcal{D}(m,n))) \neq 0$.

Also, $K_i (j * \tau_q(\mathcal{D}(m,n))) \neq 0$ and $\breve{K}_i(j * \tau_q(\mathcal{D}(m,n))) \neq 0$.

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