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Einstein's Equations of Motion in a Flat Oblate Spheriodal Space-Time

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Abstract

It is well known how to formulate and solve Einstein's equation of motion in the fields of bodies having cylindrical and spherical symmetries. But, the fact of nature is that astronomical bodies such as the Earth and sun are generally spheriodal in geometry. Therefore in this paper we derive the Einstein's equations for flat space-time in oblate spheroidal coordinates to pave the way for the investigation of the corresponding equations in the fields of bodies having spheroidal geometries.

1.0 Introduction

The oblate spheroidal coordinates of space (η, ξ, ϕ) are defined in terms of the Cartesion coordinates (x,y,z) by [1 - 3].

$$x = a(1 - \eta^2)^{1/2}(1 + \xi^2)^{1/2} \cos\phi \tag{1}$$

$$y = a(1 - n^2)^{1/2}(1 + \xi^2) \sin\phi \tag{2}$$

$$z = a\eta\xi \tag{3}$$

Where a is a constant parameter and

$$-1 \le \eta \le 1, 0 \le \xi < \infty; \quad 0 \le \phi \le 2\pi \tag{4}$$

Let X^u be the Cartesian coordinates and $\bar{X}^u$ be the corresponding oblate spheriodal coordinates of flat space-time given by

$$\left.\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \end{aligned}\right\} \tag{5}$$

and

$$\left.\begin{aligned} \bar{X}^0 &= c\bar{t} = ct \\ \bar{X}^1 &= \eta \\ \bar{X}^2 &= \xi \\ \bar{X}^3 &= \phi \end{aligned}\right\} \tag{6}$$

Then the proper time interval $d\tau$ is given in the Cartesian coordinates by

$$d\tau^2 = c^2(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{7}$$

But by the invariance of proper time [3] it follows that

$$d\bar{\tau}^2 = d\tau^2 \tag{8}$$

Where $\bar{\tau}$ is the proper time interval in the Oblate spheroidal coordinate. Also the metric tensor in the Oblate Spheroidal coordinate. Also the metric tensor in the Oblate spheroidal coordinate, $\bar{g}_{uv}$, are defined [3] by the relation

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$$\bar{g}_{uv}d\bar{X}^u d\bar{X}^v = d\bar{t}^2 \tag{9}$$

It therefore follows from (9), (8) and (7) that

$$\bar{g}_{uv}d\bar{X}^u d\bar{X}^v = c^2(dx^0)^2 - (dx^1)^2 - (dx^2)^2(dx^3)^2 \tag{10}$$

Consequently, substituting (1) – (3) into (10) we obtain the covariant metric tensor in Oblate Speroidal coordinates as

$$\bar{g}_{00} = 1 \tag{11}$$

$$\bar{g}_{11} = -\frac{a^2(\eta^2+\xi^2)}{(1-\eta^2)} \tag{12}$$

$$\bar{g}_{22} = -\frac{a^2(\eta^2+\xi^2)}{(1+\xi^2)} \tag{13}$$

$$\bar{g}_{33} = a^2(1-\eta^2)(1+\xi^2) \tag{14}$$

$$\bar{g}_{uv} = 0; \quad \text{otherwise} \tag{15}$$

Now by the reciprocal relation [3]

$$g_{uv}g^{v\lambda} = \delta_u{}^\lambda \tag{16}$$

Where $\delta_u{}^\lambda$ is the Dirac tensor. We obtain the corresponding contravariant metric tensor g^{uv} as

$$\overline{g^{00}} = 1 \tag{17}$$

$$\overline{g^{11}} = -\frac{(1-\eta^2)}{a^2(\eta^2+\xi^2)} \tag{18}$$

$$\overline{g^{22}} = -\frac{(1+\xi^2)}{a^2(\eta^2+\xi^2)} \tag{19}$$

$$\overline{g^{33}} = -a^2(1-n^2)(1+\xi^2) \tag{20}$$

$$\overline{g^{uv}} = 0; \text{ otherwise} \tag{21}$$

Now the coefficients of affine connection in the Oblate Spheroidal coordinates $\bar{\Gamma}^\alpha_{\mu\gamma}$ defined by [4]

$$\bar{\Gamma}^\alpha_{\mu\gamma} = \frac{1}{2}\bar{g}^{\alpha\lambda}\left(g_{\mu\lambda,\gamma} + g_{\lambda\gamma,\mu} - g_{\mu\gamma,\lambda}\right) \tag{22}$$

And are given by

$$\bar{\Gamma}_{11}{}^1 = \frac{\eta(1+\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)} \tag{23}$$

$$\bar{\Gamma}^1_{22} = \frac{\eta(1+\eta^2)}{(1-\xi^2)(\eta^2+\xi^2)} \tag{24}$$

$$\bar{\Gamma}^1_{33} \ \frac{\eta(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)} \tag{25}$$

$$\bar{\Gamma}^1_{12} = \bar{\Gamma}^1_{21} = \frac{\xi}{(\eta^2+\xi^2)} \tag{26}$$

$$\bar{\Gamma}^2_{22} = \frac{\xi(1+\eta^2)}{(1+\xi^2)(\eta^2+\xi^2)} \tag{27}$$

$$\bar{\Gamma}^2_{33} = \frac{-\xi(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)} \tag{28}$$

$$\bar{\Gamma}^2_{12} = \bar{\Gamma}^2_{21} = \frac{\eta}{(\eta^2+\xi^2)} \tag{29}$$

$$\bar{\Gamma}^3_{13} = \bar{\Gamma}^3_{31} = \frac{\eta}{(1-\eta^2)} \tag{30}$$

$$\bar{\Gamma}^3_{23} = \bar{\Gamma}^3_{32} = \frac{\xi}{(1+\xi^2)} \tag{31}$$

$$\bar{\Gamma}^\alpha_{\mu\gamma} = 0; \quad \text{otherwise} \tag{32}$$

Now the Einstein equations of motion in the flat space-time Oblate Spheroidal coordinates are given by [3, 4].

$$\frac{d^2\bar{X}^\alpha}{d\bar{\tau}^2}+\bar{\Gamma}^{\alpha}_{\mu v}\frac{d\bar{X}^\mu}{d\bar{\tau}}\frac{d\bar{X}^v}{d\bar{\tau}}=0 \quad (33)$$

Hence from (33) and (23) – (32) we obtain the explicit equations of motion as:

$$\ddot{t}=0$$

and (34)

$$\ddot{\eta}+\frac{\eta(1+\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2-\frac{\eta(1-\eta^2)}{(1+\xi^2)(\eta^2+\xi^2)}\dot{\xi}^2+\frac{\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi}+\frac{\eta(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)}\dot{\phi}=0 \quad (35)$$

and

$$\ddot{\xi}+\frac{\xi(1-\eta^2)}{(1+\xi)(n^2+\xi^2)}\dot{\xi}^2-\frac{\xi(1+\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2+\frac{\eta}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi}-\frac{\xi(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)}\dot{\phi}^2=0 \quad (36)$$

and

$$\ddot{\phi}+\frac{\xi}{(1+\xi^2)}\dot{\phi}\dot{\xi}+\frac{\eta}{(1-\eta^2)}\dot{\phi}\dot{\eta}=0 \quad (37)$$

These are the Einstein equations of motion in flat space-time in terms of the Oblate Spheroidal Coordinates, and referred to as the azimuthal equation.
Azimuthal Solution.
It may be noted that the azimuthal equation (37) may be written equivalently as

$$\frac{\ddot{\phi}}{\dot{\phi}}+\frac{\xi\dot{\xi}}{(1-\xi^2)}+\frac{\eta\dot{\eta}}{(1-\eta^2)}=0 \quad (38)$$

or equivalently

$$\frac{d}{dt}\left[ln\dot{\phi}\right]+\frac{d}{dt}\left[ln(1+\xi^2)^{1/2}\right]+\frac{d}{dt}\left[ln(1-\eta^2)^{-1/2}\right]=0 \quad (39)$$

Consequently the integral is given by

$$(1-\eta^2)^{-1/2}(1+\xi^2)^{1/2}\dot{\phi}=l \quad (40)$$

or equivalently

$$\dot{\phi}=(1-\eta^2)^{1/2}(1+\xi^2)^{-1/2}\,l \quad (41)$$

Where l is a constant of the motion. This result is the equation for the conservation of angular momentum for the particles in Oblate spheriodal coordinates.

2.0 Summary and Conclusions

In this paper we derived Einstein's equations of motion for a particles of nonzero rest mass in flat space-time in Oblate Spheroidal coordinates (η,ξ,ϕ) as (34) – (37). Then, we derived that exact and complete solution of the azimuthal equation as (37).

In the first place it may be noted that corresponding equation of motion for a particle of nonzero rest mass in Spherical Polar coordinates (r,θ,ϕ) are given by

$$\ddot{r}+r\dot{\theta}^2-r\dot{\phi}^2sin^2\theta=0 \quad (42)$$

and

$$r\ddot{\theta}+2\dot{r}\dot{\theta}-r\dot{\phi}^2sin\theta cos\theta=0 \quad (43)$$

and

$$r\ddot{\phi}sin\theta+2\dot{r}\dot{\phi}sin\theta+2r\dot{\theta}\dot{\phi}cos\theta=0 \quad (44)$$

Consequently, precisely as in the case of these Spherical equations the door is henceforth opened for the mathematical integration of the Oblate Spheroidal equations for physical interpretation and experimental investigation.

In the second place it may be noted that the solution of the azimuthal equation (44) corresponding to the oblate spheroidal solution (41) is given by

$$\dot{\phi}=\frac{l}{r^2sin^2\theta} \quad (45)$$

Consequently the oblate spheroidal solution (41) is a great generalization of all orders of ξ^{-2}. Now with the derivation of the Einstein's general relativistic equations of motion for a particle of nonzero rest mass in flat spheroidal space-time coordinates in this paper, the way is open for the investigation of the corresponding equation in the field around an oblate spheroidal massive body.

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