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# A New Approach to the Solution of the Nonlinear Klein-Gordon Equation

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## *Abstract*

***We present a new Modified Variational Iteration Method (MVIM) for the solution of nonlinear Klein-Gordon equations. This method is an elegant combination of the successive Taylor's approximation and the Variational Iteration Method (VIM). Numerical results show the complete reliability of the proposed technique.***

***Keywords*:** Klein-Gordon, modified variational iteration method, lagrange multiplier, Taylor's series, partial differential equation.

## 1.0 Introduction

The Klein-Gordon equation is considered one of the most important mathematical models in quantum field theory. The equation appears in relativistic physics and is used to describe dispersive wave phenomenon in general. It also appears in nonlinear optics and plasma physics. This equation appears in linear and nonlinear forms.

Several techniques including finite difference, collocation, finite element, inverse scattering, decomposition and variational iteration using Adomian's polynomials have been used to handle different types of differential equations[1-6].

The variational iteration method was proposed by J.H He [7-8]. In this paper, we present a Modified Variational Iteration Method proposed by Olayiwola [9-11] to the solution of nonlinear Klein-Gordon equation of the form:

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) + F(u(x,t)) = h(x,t) \qquad (1.0)$$

Subject to the initial condition

$$u(x,0) = f(x),\ u_t(x,0) = g(x) \qquad (1.1)$$

Where $a$ is a constant, $h(x,t)$ is a source term and $F(u(x,t))$ is a nonlinear function of $u(x,t)$.
The equation has been extensively studied by using some numerical methods.

## 2.0 Modified Variational Iteration Method (MVIM)

To illustrate the basic concept of the MVIM, we consider the following general nonlinear partial differential equation:

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \qquad (1.2)$$

where L is a linear time derivative operator, R is a linear operator which has partial derivative with respect to x, N is a nonlinear operator and g is an inhomogeneous term. According to MVIM, we can construct a correct functional as follows:

$$u_0(x,t) = u(x,0) + g_i(x)t^i \qquad (1.3)$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left[ Lu_n + R\tilde{u}_n + N\tilde{u}_n - g \right] d\tau \qquad (1.4)$$

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where $g_i(x)$ can be evaluated by substituting $u\ (x,0)$ in (1.2) and at limit $t=0, i\le 2.$ .

$\lambda$ is a Lagrange multiplier which can be identified optimally via variational iteration method. The subscript $n$ denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation i.e, $\delta\tilde{u}_n = 0$ .

## 3.0 MVIM for the solution Nonlinear Klein-Gordon Equation

We present the analytical solution of the equation of the form:

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) + F(u(x,t)) = h(x,t) \tag{1.5}$$

$$u(x,0) = f(x),\ u_t(x,0) = g(x) \tag{1.6}$$

Applying MVIM in (1.5)

$$u_{n+1} = u_n + \int_0^t (u_{n,\tau\tau}(x,\tau) - u_{n,xx}(x,\tau) + au_n(x,t) + F(u_n(x,\tau)) - h(x,\tau))\,d\tau \quad .(1.7)$$

Making (1.7) stationary, this yields the following stationary condition

$$1-\lambda'\big|_{\tau=t} = 0 \tag{1.8}$$

$$\lambda\big|_{\tau=t} = 0 \tag{1.9}$$

$$\lambda'' = 0 \tag{2.0}$$

From (1.8-2.0), we have

$$\lambda(t,\tau) = (\tau - t) \tag{2.1}$$

## 4.0 Numerical Examples

**Numerical Example 1**:- Consider nonlinear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) - u(x,t) + u(x,t)^2 = xt + x^2t^2,\ u(x,0) = 1,\ u_t(x,0) = x.$$

Applying (1.3), (1.4), (1.7) we obtain

$$g_1 = 0 \tag{2.2}$$

$$u_1 = u_0 + \left(\int_0^t (\lambda - \tau)(u_{0,\tau\tau}(x,\tau) - u_{0,xx}(x,\tau) - u_0\ (x,\tau) + u(x,\tau)^2 - x\tau - x^2\tau^2)\,d\tau\right) \tag{2.3}$$

$$u_1 = 1 + xt \tag{2.4}$$

In the same manner

$$u_2 = 1 + xt \tag{2.5}$$

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$$u_n = 1 + xt \tag{2.6}$$

MVIM admits that

$$u = \lim_{n\to\alpha} u_n \tag{2.7}$$

Which gives the exact solution

$$u = 1 + xt \tag{2.8}$$

**Numerical Example 2**:-

Consider $u_{tt}(x,t) - u_{xx}(x,t) + u(x,t)^2 = 1 + 2xt + x^2t^2,\ u(x,0) = 1,\ u_t(x,0) = x.$

Applying (1.3), (1.4), (1.7) we have

$$g_1 = 0 \quad (2.9)$$

$$u_1 = u_0 + \left( \int_0^t (\lambda - \tau)(u_{0,\tau\tau}(x,\tau) - u_{0,xx}(x,\tau) + u_0(x,\tau)^2 - 1 - 2x\tau - x^2\tau^2) d\tau \right) \quad (3.0)$$

$$u_1 = 1 + xt \quad (3.1)$$

$$u_2 = 1 + xt \quad (3.2)$$

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$$u_n = 1 + xt \quad (3.3)$$

As $n \to \infty$

$$u = \lim_{n \to \alpha} u_n \quad (3.4)$$

Which gives the exact solution

$$u = 1 + xt \quad (3.5)$$

**Numerical Example 3**:-

Consider $u_{tt}(x,t) - u_{xx}(x,t) + u(x,t)^2 = 6xt\ (x^2 - t^2)$ , $u(x,0) = 1, u_t(x,0) = 0.$

Applying (1.3), (1.4), (1.7), we obtain

$$g_3 = x^3 \quad (3.6)$$

$$u_1 = u_0 + \left( \int_0^t (\lambda - \tau)(u_{0,\tau\tau}(x,\tau) - u_{0,xx}(x,\tau) + u_0(x,\tau)^2 - 6x\tau( x^2 - \tau^2)) d\tau \right) \quad (3.7)$$

$$u_1 = x^3 t^3 \quad (3.8)$$

$$u_2 = x^3 t^3 \quad (3.9)$$

$$u_3 = x^3 t^3 \quad (4.0)$$

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$$u_n = x^3 t^3 \quad (4.1)$$

Taking the limit

$$u = \lim_{n \to \alpha} u_n \quad (4.2)$$

Therefore, (4.0) is the exact solution.

$$u = x^3 t^3 \quad (4.3)$$

**Numerical Example 4**:-

Consider $u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) + u(x,t)^2 = x^2 \cos^2(t)$ , $u(x,0) = x, u_t(x,0) = 0.$

Applying (1.3), (1.4), (1.7), we obtain

$$g_1 = -\frac{x}{2} \quad (4.4)$$

$$u_1 = u_0 + \left( \int_0^t (\lambda \; - \; \tau)(u_{0,\tau\tau}(x,\tau) - u_{0,xx}(x,\tau) + u_0 \; (x,\tau) + u_0(x,\tau)^2 - x^2\cos^2(\tau) \;\;)d\tau \right) \quad (4.5)$$

$$u_1 = x - \frac{x}{2}t^2 + \frac{x}{24}t^4 - \frac{x}{720}t^6 + ... = x\cos(t) \quad (4.6)$$

Hence, in a closed form $u = x\cos(t)$.

## 4.0 Conclusion

In this paper, the modified variational iteration methods has been successfully applied to various forms of the nonlinear Klein-Gordon equation. We showed that the results obtained converged to the exact solution after one or two iterations.

The method is applied in a direct way without using linearization, transformation, perturbation, discretization or restrictive assumptions. The use of Taylor's successive approximation gives this method a clear advantage over the other methods because it reduces the more successive application of integral operator.

It worth to note, that the method is elegant with minimal computational efforts. This method can be extended to the sine-Gordon equations.

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