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Computational results of Advection equation using FTBS scheme.

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Abstract

We consider the one dimensional equation that governs the motion of a conserved scalar as it is advected by a known velocity field. We use the FTBS or forward in time and backward in space scheme for the Advection equation. If expanding the numerical scheme by Taylor series expansion and truncating it after the first two terms, we found that, the discretized equation satisfied the partial differential equation. A further result that admits the theorem of Lax, shows that the numerical properties of consistency, stability and convergence are satisfied.

Keywords: Advection equation, numerical scheme, Damped radio wave, constant velocity, consistency, stability, convergence and Lax theorem.

1.0 Introduction

We focus on the Advection equation

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} \tag{1}$$

Here, u=u(x,t) and c=c(x,t) in which x is space and t is time. The Advection equation is hyperbolic partial differential equation that governs the motion of a conserved scalar as it is advected by a known velocity field [1].
For example the Advection equation can be considered in the form of a damped radio wave [2] or it can be applied to the transport of dissolved salt in water.
Even in one space dimension and constant velocity the system remain difficult to solve. Since the advection equation is difficult to solve numerically, interest typically centers on discontinuous shock solutions, which are notoriously hard for numerical schemes to handle [3].
Using the FTBS or forward in time and backward in space scheme, given as

$$u_j^{n+1} = u_j^n - \alpha\left(u_j^n - u_{j-1}^n\right) \tag{2}$$

where α is some constant [4], and also, from a Taylor series expansion, it is shown that scheme (2) is first order accurate in space and time [5]. We have,

$$u_j^{n+1} = u_j^n - \alpha\left(u_j^n - u_{j-1}^n\right)$$

$$u_j^{n+1} = u_j^n + \Delta t u_t^n + \frac{\Delta t^2}{2}u_{tt}^n + \cdots$$

$$u_j^n + \Delta t u_t^n + \frac{\Delta t^2}{2}u_{tt}^n = u_j^n - \alpha u_j^n + \propto u_{j-1}^n \tag{3}$$

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$$u_j^n + \Delta t u_t^n + \frac{\Delta t^2}{2} u_{tt}^n = u_j^n - \alpha u_j^n + \alpha u_j^n - \alpha \Delta x u_x^n + \alpha \Delta x^2 u_{xx}^n \quad (4)$$

$$\Delta t u_t^n + \frac{\Delta t^2}{2} u_{tt}^n = \alpha \Delta x u_x^n + \alpha \Delta x^2 u_{xx}^n \quad (5)$$

$$\Delta t u_t^n = \alpha \Delta x u_x^n + \alpha \frac{\Delta x^2}{2} u_{xx}^n - \frac{\Delta t^2}{2} u_{tt}^n \quad (6)$$

$$\Delta t u_t^n = \alpha \Delta x u_x^n + O(\alpha \Delta x^2 - \Delta t^2) \quad (7)$$

The truncation error on the right hand side is of the first order in both time and space as limit $\Delta t \; and \; \Delta x$ tends to zero. Thus the forward in time and backward in space scheme satisfies the Advection equation (1). We can also investigate the behaviour of scheme (2) for the initial condition in the form of a square wave given by

$$\left.\begin{array}{l} f(x) = 0, \; x < 0.1 \\ f(x) = 1, \; 0.1 \leq x \leq 0.2 \\ f(x) = 0, \; x > 0.2 \end{array}\right\} \quad (8)$$

and calculate also the numerical solution of the convection equation (1).
Using the forward in time and backward in space scheme, the exact solution of equation (1) $\bar{u}$ is one of pure translation [6], that is,

$$\bar{u}(x,t) = f(x-t) \quad (9)$$

We now give a method which deal on the accuracy of a numerical solution compared to that of the exact solution of the differential equation.

Equivalence theorem of Lax [7]:
For a well posed linear initial value problem with a consistent decretization, stability is the necessary and sufficient condition for convergence of the numerical scheme.
We now discuss these properties of the numerical scheme one by one.

2.0 Properties of numerical Scheme

1.1 Consistency:

A formal way of demonstrating consistency of a numerical scheme is to determine the function error by performing a Taylor series expansion and to show that it reduces to zero as Δx and Δt tend to zero. Using this method, it has been shown that, the FTBS scheme for the one-dimensional wave equation is consistent.

1.2 Analysis of stability

Consider the one-dimensional convection equation (1) and its discretization using the FTBS scheme given by equation (2).
Let the analytical solution of the partial differential equation (1) be denoted by M, the exact solution of the discretized equation (2) be denoted by N, and the numerical solution of the discretized equation (2) obtained with finite machine accuracy be denoted by S.
Then we can write

Discretization error =M-N (10)

Round-off error =S-N (11)

Since stability deals with accuracy with which the computed solution S approaches the exact solution N of the discretized equation, we can define error η as

$$\eta = N - S \text{ or } S = N - \eta \quad (12)$$

Using the superscript n for the finite step and the subscript i for the spatial location so that f(x,t) can be written as $f(i\Delta x, n\Delta t)$ or as f_i^n, we can denote the error at i^{th} space location and n^{th} time step as η_i^n.
Since the computed solution must satisfy the discretized equation (2), we have to machine accuracy,

$$(S_i^{n+1} - S_i^n)\Delta t - \frac{c(S_i^n - S_{i-1}^n)}{\Delta x} = 0 \quad (13)$$

Substitute $S_i^n = N_i^n - \eta_i^n$, in equation (13), we obtain

$$\left(N_i^{n+1} - \eta_i^{n+1}\right) - \frac{(N_i^n - \eta_i^n)}{\Delta t} - c\left(N_i^n - \eta_i^n\right) - \frac{(N_{i-1}^n - \eta_{i-1}^n)}{\Delta x} = 0$$

Or

$$\frac{\left(N_i^{n+1} - N_i^n\right)}{\Delta t} - \frac{c(N_i^n - N_{i-1}^n)}{\Delta x} + \frac{\left(\eta_i^{n+1} - \eta_i^n\right)}{\Delta t} - \frac{c(\eta_i^n - \eta_{i-1}^n)}{\Delta x} = 0 \qquad (14)$$

Thus, the error equation and the computed solution of the discretized equation both posses the same form and the same growth property in time.
Thus for stability, we should have

$$\left|\frac{\eta_i^{n+1}}{\eta_i^n}\right| \leq 1 \qquad (15)$$

1.3 Convergence

Since the numerical scheme is consistent and stable, then by theorem, the scheme converges.

3.0 Numerical experiment

Example 3.1
Solve the partial differential equation

$u_t = u_x$
u(0,t)=0
u(1,t)=2
u(x,0)=2x
at the point x=i: i=0,1,2,3,…,7.
and t=j/8: j=0,1,2,3,4.
Solution
$c^2=2$, h=1, k=1/8
$a=c^2k/h^2 = \frac{2\times 1}{1\times 8} = \frac{1}{4}$
Then the equation is
$u_{i,j+1}=1/4(u_{i-1,j}+u_{i+1,j})$
$u_{0,1}=0$, $u_{1,0}=2$, $u_{2,0}=4$, $u_{3,0}=6$, $u_{4,0}=8$, $u_{5,0}=10$, $u_{6,0}=12$, $u_{7,0}=14$
$u_{i,j+1}=1/4(u_{i-1,j}+u_{i+1,j})$
j=0, $u_{1,1}=1/4(u_{0,0}+u_{2,0})$= 1/4(0+4) = 1.00
$u_{2,1}=1/4(u_{1,0}+u_{3,0})$ =1/4(2+6) =8/4 =2.00
$u_{3,1}=1/4(u_{2,0}+u_{4,0})$ =1/4(4+8) =12/4 =3.00
$u_{4,1}= 1/4(u_{3,0}+u_{5,0})$ =1/4(6+10) =16/4 =4.00
$u_{5,1}= 1/4(u_{4,0}+u_{6,0})$ =1/4(8+12) =20/4 =5.00
$u_{6,1}= 1/4(u_{5,0}+u_{7,0})$ = 1/4(10+14) = 24/4 =6.00
$u_{7,1}=1/4(u_{6,0}+u_{8,0})$ = 1/4(12+16) =28/4 =7.00
j=1, $u_{1,2}=1/4(u_{0,1}+u_{2,1})$ = 1/4(0+2)=2/4 = 0.50
$u_{2,2}=1/4(u_{1,1}+u_{3,1})$ =1/4(1+3) =4/4 =1.00
$u_{3,2}=1/4(u_{2,1}+u_{4,1})$ =1/4(2+4) = 6/4 =1.50
$u_{4,2}=1/4(u_{3,1}+u_{5,1})$ = 1/4(3+5) = 8/4 =2.00
$u_{5,2}=1/4(u_{4,1}+u_{6,1})$ = 1/4(4+6) =10/4 =2.50
$u_{6,2}=1/4(u_{5,1}+u_{7,1})$ = 1/4(5+7) =12/4 = 3.00
$u_{7,2}=1/4(u_{6,1}+u_{8,1})$ =1/4(6+0) = 6/4 =1.50
j=2, $u_{1.3}= 1/4(u_{0,2}+u_{2,2})$ = 1/4(0+1) = 1/4 =0.25
$u_{2,3}=1/4(u_{1,2}+u_{3,2})$ = 1/4(0.5+1.5) =2/4 =0.50
$u_{3,3}= 1/4(u_{2,2}+u_{4,2})$ =1/4(1+2) =3/4 = 0.75
$u_{4,3}= 1/4(u_{3,2}+u_{5,2})$ = 1/4(1.5+2.5) = 4/4 =1.00
$u_{5,3}= 1/4(u_{4,2}+u_{6,2})$ =1/4(2+3) = 5/4 = 1.25
$u_{6,3}= 1/4(u_{5,2}+u_{7,2})$ =1/4(1.5+2.5) =4/4 = 1.00
$u_{7,3}= 1/4(u_{6,2}+u_{8,2})$ = 1/4(3+0) = 3/4 = 0.75
j=3, $u_{1,4}= 1/4(u_{0,3}+u_{2,3})$ = 1/4(0+0.5) =0.5/4 =0.12

$u_{2,4} = 1/4(u_{1,3}+u_{3,3}) = 1/4(0.25+0.75) = 1/4 = 0.25$
$u_{3,4} = 1/4(u_{2,3}+u_{4,3}) = 1/4(0.5+1) = 1.5/4 = 0.38$
$u_{4,4} = 1/4(u_{3,3}+u_{5,3}) = 1/4(0.75+1.25) = 2/4 = 0.50$
$u_{5,4} = 1/4(u_{4,3}+u_{6,3}) = 1/4(1+1) = 2/4 = 0.50$
$u_{6,4} = 1/4(u_{5,3}+u_{7,3}) = 1/4(1.25+0.75) = 2/4 = 0.50$
$u_{7,4} = 1/4(u_{6,3}+u_{8,3}) = 1/4(1+0) = 1/4 = 0.25$
$j=4,\ u_{1,5} = 1/4(u_{0,4}+u_{2,4}) = 1/4(0+0.25) = 0.25/4 = 0.061$
$u_{2,5} = 1/4(u_{1,4}+u_{3,4}) = 1/4(0.12+0.38) = 0.5/4 = 0.12$
$u_{3,5} = 1/4(u_{2,4}+u_{4,4}) = 1/4(0.25+0.50) = 0.75/4 = 0.19$
$u_{4,5} = 1/4(u_{3,4}+u_{5,4}) = 1/4(0.38+0.5) = 0.88/4 = 0.22$
$u_{5,5} = 1/4(u_{4,4}+u_{6,4}) = 1/4(0.5+0.5) = 1/4 = 0.25$
$u_{6,5} = 1/4(u_{5,4}+u_{7,4}) = 1/4(0.5+0.25) = 0.75/4 = 0.19$
$u_{7,5} = 1/4(u_{6,4}+u_{8,4}) = 1/4(0.5+0) = 0.5/4 = 0.12$

Table 3.1. Solution to Example 3.1

j \ i	0	1	2	3	4	5	6	7
0	0	2.0	4.0	6.0	8.0	10.0	12.0	14.0
1	0	1.0	2.0	3.0	4.0	5.0	6.0	7.0
2	0	0.50	1.0	1.50	2.0	2.50	3.0	1.50
3	0	0.25	0.50	0.75	1.0	1.25	1.0	0.75
4	0	0.12	0.25	0.38	0.5	0.5	o.5	0.25
5	0	0.11	0.12	0.26	0.24	0.28	0.24	0.12

4.0 Conclusion:

Table 3.1 shows that the numerical solutions satisfy the consistency, stability and convergence principle. It then follows that the FTBS scheme approximates the solutions for the advection equation (1) as seen in Table 3.1.

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