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Numerical Methods for Solution of Initial Value Problem Using Sylvester's Sequence and Egyptian Fractions

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Abstract

Sylvester's Sequence and Finite Egyptian Fraction seem to be uniformly, rapidly decreasing values of fractions which converge to the unit number one. We examine and utilize this property to develop numerical methods for the solution of initial value problems.

***Keywords*:** Sylvester's sequence, Egyptian fractions, initial value problems, finite linear multi-step scheme, convergence, zero stability and consistency

1.0 Introduction

Sylvester's sequence, according to Orobosa , is given by the formula;

$$S_{n+1} = 1 + \prod_{i=0}^{n} S_i \qquad (1)$$

Where $S_0 = 1$;

Thus

$$S_1 = 1 + S_0 = 2$$

$$S_2 = 1 + \prod_{I=0}^{1} S_i = 1 + S_0 * S_1 = 1 + 1 * 2 = 3$$

$$S_3 = 1 + S_0 * S_1 * S_2 = 7$$

The unit fractions formed by the reciprocals of the values in Sylvester's Sequence generate the infinite series

$$\sum_i \frac{1}{S_i} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \ldots$$

Which is the Egyptian fractional representation of the unit number one[1], that is,

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \ldots \qquad (2)$$

Thus, we can find the finite Egyptian fractional representations of (1) of any length by truncating the series (2) and subtracting the sum from (1) to form the last fractional [2].

Thus: for j = 2 and 3 we have the finite sums

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$$\left.\begin{aligned} 1 &= \frac{1}{2}+\frac{1}{3}+\frac{1}{6} \\ 1 &= \frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{42} \end{aligned}\right\} \qquad (3)$$

2.0 The Finite Difference Linear Multistep Methods

We consider the linear multistep method of the form:

$$\sum_{j=0}^{k} \alpha_j \; y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} \qquad (4)$$

The first and second characteristic polynomials of the linear multistep method are respectively;

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$$

and

$$\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j$$

For consistency, we must have

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1) \qquad (5)$$

(see [3] and [4]).

It follows from (5) that we can have a linear multistep method of order k, where the β_j's, $j = 0,1,2,...k$ are the finite Egyptian fractional representation of (1) specifically for K = 2, we have

$$y_{n+2} - y_{n+1} = \left(\frac{1}{6} f_{n+2} + \frac{1}{3} f_{n+1} + \frac{1}{2} f_n \right)$$

i. e.

$$y_{n+2} - y_{n+1} = \frac{h}{6}\left(f_{n+2} + 2f_{n+1} + 3f_n\right) \qquad (6)$$

$$y_{n+2} - y_n = \frac{h}{3}\left(f_{n+2} + 2f_{n+1} + 3f_n\right) \qquad (7)$$

The methods (6) and (7) are consistent since they satisfy the condition (5) above

3.0 Convergence of the methods

Theorem: The numerical methods (6) and (7) are convergent

Proof:

The necessary and sufficient conditions for a linear multistep method to be convergent are that it be consistent and zero-stable.

(a) Consistency: The methods are consistent since they satisfy condition (5) above

(b) Zero-Stability: By Lambert , a linear multistep method is said to be zero stable if no root of the first characteristic polynomial $\rho(\xi)$ has modulus not greater than one and if every root with modulus one is simple.

Now, for methods (6) and (7) respectively,

(i) $\rho(\xi) = \xi^2 - \xi$ *and*

(ii) $\rho(\xi) = \xi^2 - 1$

Both (i) and (ii) satisfy the zero-stability criteria.

It follows therefore that the methods (6) and (7) are convergent.

4.0 Numerical Experiments

Example 4.1

$$y' = x + y; \quad x_0 = 0, \; y_o = 1$$

Table 4.1. Solution to Example 4.1

X	Exact solution	Approximate solution	
		h = 0.1	h = 0.01
0.1	1.110341836	1.110341836	1.109613613
0.2	1.242805516	1.228299309	1.241195991
0.3	1.399717616	1.382683399	1.397049588
0.4	1.583649396	1.548249792	1.57971817
0.5	1.797442542	1.756036267	1.7920121
0.6	2.0442376	1.97945916	2.037036282
0.7	2.327505414	2.252003969	2.318221037
0.8	2.651081856	2.545729662	2.639356198
0.9	3.019206222	2.896813457	3.004628797
1.0	3.436563656	3.275955369	3.41866469

Example 4.2

$$y' = 4x + 2y; \quad x_0 = 0, \; y_o = 1$$

Table 4.2. Solution to Example 4.2

X	Exact solution	Approximate solution	
		h = 0.1	h = 0.01
0.1	1.242805516	1.242805516	1.239608892
0.2	1.583649396	1.5204	1.575845041
0.3	2.0442376	1.957942858	2.029947337
0.4	2.651081856	2.46307755	2.627823015
0.5	3.436563656	3.183509039	3.401073787
0.6	4.440233846	4.021600999	4.388247481
0.7	5.710399934	5.153311765	5.636364817
0.8	7.306064848	6.478245823	7.20278177
0.9	9.29929493	8.208293102	9.15746127
1.0	11.7781122	10.24464365	11.58574424

5.0 Discussion of Results and Conclusion

Tables 4.1 and 4.2 show that the methods (6) and (7) are convergent and provide good approximations for the solution of the problems to which they are applied

With these, we have shown that Sylvester's sequence and Egyptian fractions are quite suitable to be used for finite difference schemes for the solution of initial value problem and it is hoped that more people will show interest in them and try to investigate further.

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