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Heat Flow in a PTC Thermistor with an Exponential Function Conductivity

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Abstract

The mathematical model for the description of heat conduction in a thermistor is a coupled system of nonlinear partial differential equations. This paper is on the numerical solution of this model with an exponential function electrical conductivity with a view to analyse the flow of heat in the device. Numerical results are found to be in good agreement with existing results and represent physical characteristics of the thermistor.

1.0 Introduction

The thermistor is a ceramic thermo-electric device which may form one element of an electric circuit. The basis for its conductivity is an electrical conductivity that is highly a nonlinear function of temperature [1]. The typical thermistor model is an initial boundary value problem comprising of coupled nonlinear differential equation for heat and current flow [2, 3]. Many researchers have worked on this field: Wiedman [4], under the title 'The thermistor problem' proved the existence of a classical solution to the problem. Howson et al [5] gave stationary solutions to the problem. Results on existence and uniqueness of solutions weregiven by Antontsevand Chipot[6]. Fowler et al [7], studied the heat transfer of the device under the title "Temperature surges in a current limiting circuit device", developing approximate methods to describe the transient heat flow which occurs when the current is switched on. Wood and Kutluay [1] applied the heat balance integral method to solve the problem.Kutluay and Wood [8] considered the heat flow through a PTC thermistor and presented approximate steady state solution of the problem with a ramp electrical conductivity using a standard explicit finite difference method. In their conclusion they stated that a ramp function is not a particularly good model for conductivity since it is a stretched form of the step one. They recommended that a model of exponential function conductivity be investigated, hence this work.

This paper analyses the heat flow in the PTC thermistor with exponential function electrical conductivity using the method of lines. Results obtained for cold, warm and hot phases are found to exhibit physical characteristics of a PTC thermistor.

In the rest of the paper, the PTC thermistor model is presented in section two of the paper. The method of lines solution is presented in section three, section four is on results presentation and discussion and conclusion is found in section five of the paper. The list of references concludes the paper.

2.0 Problem Statement

The typical thermistor model is an initial-boundary-value problem comprising of coupled non-linear differential equations for heat and current flow. The dimensionless temperature satisfies the heat equation [1, 2]

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha\sigma\left(\frac{\partial \phi}{\partial x}\right)^2 \quad , \quad 0<x<1, \;\; t>0 \tag{1}$$

subject to boundary and initial conditions

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$$\frac{\partial u}{\partial x}=0\ ,\ x=0,\ t>0\ , \tag{2}$$

$$\frac{\partial u}{\partial x}+\beta u=0\ ,\ x=1,\ t>0 \tag{3}$$

and initial condition

$$u(x,0)=0\ ,\ \ 0\le x\le 1 \tag{4}$$

in which β is a positive heat transfer coefficient and α is the ratio of electric heating to heat diffusion.

The electric potential $\phi(x,t)$ is governed by

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial\phi}{\partial x}\right)=0\quad ,\quad 0<x<1,\ \ t>0 \tag{5}$$

Subject to

$$\phi(0,t)=0,\ t>0\ ,\qquad \phi(1,t)=0,\ t>0 \tag{6}$$

and initial condition

$$\phi(x,0)=x,\ 0\le x\le 1 \tag{7}$$

The electric conductivity is assumed to be of the form

$$\sigma(U)=\begin{cases}1 & 0\le U\le 1\\ \delta & U\ge 1\end{cases} \tag{8}$$

wheretypically $\delta=10^{-5}$

According to Kutluay et al [1, 2], assuming monotonicity of the temperature profile such that the point $x=0$ will always be the hottest and will be the first point to reach the critical temperature $U_C=1$above which σ drops. Then,due to the decrease in σ, the rate of heat loss at $x=1$will ultimately equal the internal heat generationand a steady-state will be reached, which may be any one of the three phases described below [1, 7]:

• **Cold phase:** $0<t\le t_0$

In this phase $0<U(x,t)\le U_c$ and $\sigma(U)=1$. The requirements $U(0,t)>U(1,t)$ for $t\in(0,\ t_0]$ and $U(0\ ,t_0)=U_c$ should be satisfied.

• **Warm phase:** $t_0<t\le t_1$

In this phase a hot region with $U(x,t)>U_c$ and $\sigma(U)=\delta$ is separated from the a region with $U(x,t)\le U_c$ and $\sigma(U)=1$ by a moving boundary $0<s\le 1$ on which $U=U_c$. The requirements $U(0,t)>U(1,t)$ for $t\in(t_0\,,\,t_1]$ and $U(1,t_1)=U_c$ should be satisfied.

Hot Phase $t>t_1$

In this phase $U(x,t)>U_c$ and $\sigma(U)=\delta$. The requirement $U(0,t)>U(1,t)>U_c$ for $t>t_1$ should be satisfied.

In this paper we adopt the exponential function electrical conductivity [1, 7, 8], expressed as

$$\sigma=\exp[-f(u)/\varepsilon],\ \varepsilon\approx 10^{-1}\ \text{where}$$

$$f(u)=\begin{cases}0 & 0\le u\le 1\\ u-1 & 1<u<2\\ 1 & u\ge 2\end{cases} \tag{9}$$

Taking $\varepsilon=\dfrac{0.2}{\ln 10}(\approx 0.087)$

The exact solution of the electric potential problem (5), (6) and (7) can be easily found to be

$$\phi(x,t) = x \ (0 \le x \le 1 \text{ and } t \ge 0)$$

and the thermistor problem can be described thus:

For the cold Phase, theelectrical conductivity is described by $\sigma(u) = 1$, thus the thermistor problem is reduced to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha \quad , \qquad 0 < x < 1, \quad t > 0 \tag{10}$$

Supplemented by conditions (2) and (3).

For the warm phase,theelectrical conductivity is described by

$$\sigma(u) = \exp\left(\frac{-(u_1 - 1)}{0.087}\right),$$

Therefore, the thermistor problem for the warm phase is reduced to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha \exp\left(\frac{-(u_1 - 1)}{0.087}\right) \quad , \qquad 0 < x < 1, \quad t > 0 \tag{11}$$

Supplemented by conditions (2) and (3).

For the hot phase, theelectrical conductivity is described by $\sigma(u) = \delta$, therefore the thermistor problem is reduced to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha\sigma \quad , \qquad 0 < x < 1, \quad t > 0 \tag{12}$$

Supplemented by conditions (2) and (3).

3.0 Method of Line Solution

The method of lines (MOL) is a well established numerical technique or rather a semi analytical method for the analysis partial differential equation models.It is regarded as a special finite difference method but more effective with respect to accuracy and computational time than the regular finite difference method. The method of lines (MOL) involves discretising the spatial domain and thus replacing the partial differential equation with a vector system of ordinary differential equations (ODEs) [9].Efficient and effective integrating packages have been developed for solving ordinary differential equations [10, 11]. The MATLAB package, for example, has strong vector and matrix handling capabilities, a good set of ODE solvers, and an extensive functionality which can be used to implement the MOL [10, 11]. MOL has the merits of both the finite difference method and analytical methods. Results on stability of the method are given by [12, 13].

Using the usual central difference approximation for $\frac{\partial^2 u}{\partial x^2}$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + O(\Delta x^2)$$

Substituting in (11) gives

$$\frac{\partial u_i}{\partial t} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + \alpha \exp\left(\frac{-(u_i - 1)}{0.087}\right) \tag{13}$$

The second order approximation for u_x is given as

$$u_x = \frac{u_{i+1} - u_{i-1}}{2(\Delta x)} + O(\Delta x^2)$$

Applying this to the boundary condition (2) we have

$$u_{i+1} = u_{i-1} \qquad i = 1 \tag{14}$$

And to the boundary conditions (3) we have

$$u_{i+1} = u_{i-1} - 2\beta \Delta x u_i, \quad i = N \tag{15}$$

Substituting(14) and (15) into (13) gives a system of approximating ordinary differential equations

$$\dot{u}_1 = \frac{1}{(\Delta x)^2}\left(-2u_1 + 2u_2\right) + \alpha \exp\left(\frac{-(u_1 - 1)}{0.087}\right)$$
$$\dot{u}_2 = \frac{1}{(\Delta x)^2}\left(u_1 - 2u_2 + u_3\right) + \alpha \exp\left(\frac{-(u_2 - 1)}{0.087}\right)$$
$$\dot{u}_3 = \frac{1}{(\Delta x)^2}\left(u_2 - 2u_3 + u4\right) + \alpha \exp\left(\frac{-(u_3 - 1)}{0.087}\right)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\dot{u}_N = \frac{1}{(\Delta x)^2}\left(-2u_{N-1} + 2u_N\right) + \alpha \exp\left(\frac{-(u_N - 1)}{0.087}\right) \qquad (16)$$

whichis a tridiagonal system of algebraic equations with initial condition

$$u_i(0) = 0 \qquad (17)$$

This system of ordinary differential equations (ODEs) is then integrated using the Matlab integrator ode15s which is a stiff integrator

1.0 Results and Discussions

The values of α and β are chosen in agreement with that used in [1],obtained from the steady state solution. For cold phase, $\alpha = 0.1,\ \beta = 0.2$ are used, for warm phase, $\alpha = 0.5,\ \beta = 0.2$ and for hot phase, $\alpha = 10^5,\ \beta = 0.2$ are used. Method of lines results for each phase is given on Table 1. The evolution of temperature for each phase is shown by Figures 1 to 3.

Table 1 Method of lines solution for cold, hot and warm phases

x	$u(x)$ Cold Phase	$u(x)$ Warm Phase	$u(x)$ Hot Phase
0.0	0.550000	1.1081261	5.50000
0.1	0.549500	1.1074051	5.49500
0.2	0.548000	1.0522810	5.48000
0.3	0.545500	1.1015601	5.45500
0.4	0.542000	1.0963361	5.42000
0.5	0.537500	1.0894601	5.37500
0.6	0.532000	1.0807961	5.32000
0.7	0.525500	1.0701561	5.25500
0.8	0.518000	1.0572841	5.18000
0.9	0.509500	1.0418231	5.09500
1.0	0.500000	1.0232710	5.00000

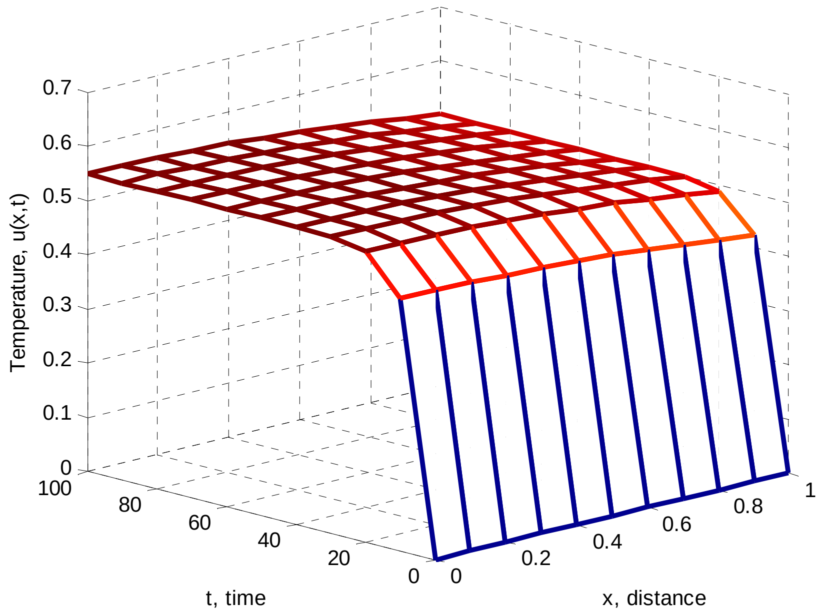

Figure 1 Temperature evolutions in the cold phase.

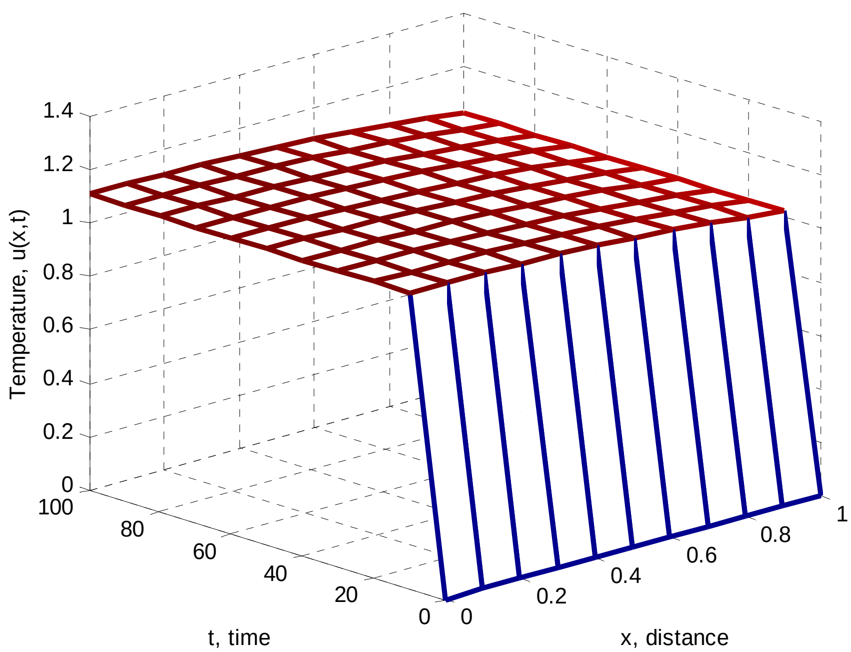

Figure 2 Temperature evolutions in the warm phase.

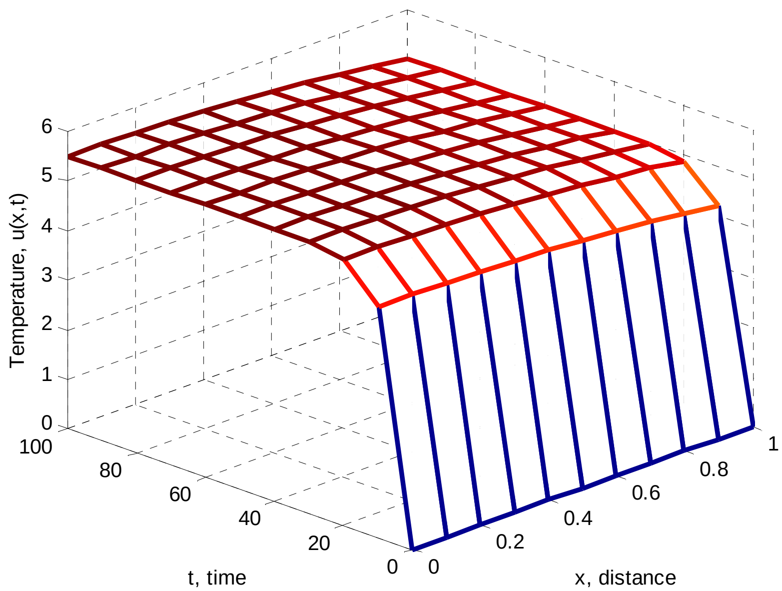

Figure 3 Temperature evolutions in the warm phase.

2.0 **Conclusion**

We have used the method of lines to show the analysis of heat flow in a PTC thermistor with exponential function conductivity. Results from the table display the physical behavior of a thermistor in the cold, warm and hot phases. We therefore conclude that the exponential function is a good model for the electrical conductivity and that the method of lines is a good method for the solution of the thermistor problem.

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