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New Iterative Schemes for Solving Nonlinear Equations

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Abstract

In this paper, we present twonew iterative schemes for solving problems of nonlinear equations from the classical Taylor's series method. The methods are constructed by applyingthe Adomian decomposition method and are compared with other iterative methods using a two wayanalysis of variance (ANOVA).They were found to be very efficient and better than some of the existing schemes. Some numerical examples are given to justify the efficiency of the new iterative schemes.

Keywords**:** Nonlinear equations, iterative methods, Adomian decomposition method.

1.0 Introduction

Finding the easiest and convenient way of solving problemsof nonlinear equations is very importantin science and engineering. This problem is also termed as root finding problem. The value of x which satisfies $f(x) = 0$ is called root of $f(x)$ and more often it is called root at zero of $f(x)$, where $f(x)$ is a continuously differentiable real or complex function.Besides polynomial equations, there are many problems in science and engineering applications that involve the function of transcendental and exponential nature. Most often, numerical methods are used to obtain the approximate solution of such problems.Newton (or Newton-Raphson) method is probably the most widely used algorithm for finding simple roots[1], which starts with an initial approximation x_0 closer to the root x and generates a sequence of successive iterates $\{x_k\}_{k=0}^{\infty}$ converging quadratically to simple roots.A new method for solving nonlinear functional equations of all kinds was proposed byAdomian [2] which is now well known as the Adomian decomposition method. This method has been applied to various problems both deterministic and stochastic, linear and nonlinear arising in science and engineering.The method decomposes the nonlinear part as a series function.The Adomian decomposition method tackles the problem directly and in a straightforward fashion without using linearization or any other restrictive assumptions [3].The series converges fast to the exact solution, if there is a single solution possible, and to one of the possible solutions, if several solutions exist [2].

Abbasbandy [4] proposed an iterative method for improving the Adomian decomposition method. The scheme is based on both Adomian decomposition method and Newton-Raphson method.The scheme for the method is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x)}{2[f'(x_n)]^3} - \frac{[f(x_n)]^3 [f''(x_n)]^2}{2[f'(x_n)]^5}$$

Basto et al[5] developed a third-order convergence method, which is also based on Newton-Raphson method and Adomian decomposition method and considering the Taylor's series expansion around x. Basto's iterative scheme is given as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x)}{2[f'(x_n)]^3 - 2f(x_n)f'(x_n)f''(x_n)}$$

In this paper, we develop two new iterative methodsfor solving nonlinear equations. Both methodsconverge cubically and do not require the computation ofsecond derivatives.These methods are derived by considering the Taylor's series expansion around x of higher order andthen applying Adomian decomposition method.The new iterative methodsare compared with other existing methods such as Newton-Raphson, Adomian's decomposition method, Abbasbandy's method and Basto et al method.The proposed methods behave equally or better than some of the existing schemes.

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2.0 Methodology used in this study:

We begin by stating the Taylor's theorem as follows.

Taylor's Theorem:Suppose $f \in c^n[a,b]$, that f^{n+1} exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $\xi(x)$between x_o and x with

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n + R_n(x)$$

Where $R_n(x) = \frac{f^{n+1}\xi(x)}{(n+1)!}(x - x_0)^{n+1}$ (called theremainder term or truncation error).

By taking the first two terms of the series and ignoring others,the Newton-Raphson method was developed and is given as follows: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ where $x_{n+1}$is the new iterate and x_n is the pervious iterate.

The Adomian approach;

The decomposition method using Adomian polynomials was used to solve different problems in applied mathematics in [6].The method is outlined as follows:

consider $Fy = f$

where F is a nonlinear differential operator and both y and f are functions of t. Rewriting the equation in operator form we get

$$Ly + Ry + Ny = f \qquad (1)$$

where*L* is an operator representing the linear portion of *F* which is easily invertible [2], *R*is the remainder of the linear portion, and *N* is a nonlinear operatorrepresenting the nonlinear terms in *F*. Applying the inverse operator L^{-1}, equation (1) becomes

$$L^{-1}Ly = L^{-1}f - L^{-1}Ry - L^{-1}Ny .$$

Since *F* was taken to be a differential operator and *L* is linear, L^{-1} would represent integration and with any given initial or boundary conditions, $L^{-1}Ly$ will give an equationfor *y* incorporating these conditions[2]. This gives

$$y(t) = g(t) - L^{-1}Ry - L^{-1}Ny .$$

Where $g(t)$ represents the function generated by integrating *f* and using the initial/boundary conditions. Assume that the unknown function can be written as an infinite series

$$y(t) = \sum_{n=0}^{\infty} y_n(t)$$

We set $y_0 = g(t)$ and the remaining terms are to be determined by a recursive relationship defined below. This is found by first decomposing the nonlinear term into a series of Adomian polynomials, A_n. The nonlinear term is written as

$$Ny = \sum_{n=0}^{\infty} A_n$$

where A_n are functions called Adomian polynomials depending on $y_0, y_1, \cdots, y_n$, [4]

To determine the Adomian polynomials, a grouping parameter, λ is introduced as follows:

$$y(t) = \sum_{n=0}^{\infty} \lambda^n y_n$$

and

$$Ny(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$$

Then the A_n's can be generated by using the expression

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, n = 0,1,2,\cdots \quad (2)$$

Itshould be noted that λ is not a "smallness parameter" [2].
The first few polynomials for all kind of nonlinearity [2] are given as

$$A_0 = f(y_0)$$
$$A_1 = y_1 f'(y_0)$$
$$A_2 = y_2 f'(y_0) + \frac{1}{2}y_1^2 f''(y_0)$$
$$A_3 = y_3 f'(y_0) + y_1 y_2 + \frac{1}{3!}y_1^3 f'''(y_0)$$
$$\vdots$$

3.0 Construction of the new schemes

3.1*New iterative scheme 1*

Consider the Taylor's series expansion around x, with $x - x_0 = h$, to obtain

$$f(x-h) = f(x) - hf'(x) + h^2\frac{f''(x)}{2!} - h^3\frac{f'''(x)}{3!} + \cdots + h^n\frac{f^n(x)}{n!} + \cdots$$

Truncating the higher order values of the series from the third order and equating to zero, we obtain

$$f(x) - hf'(x) + \frac{h^2}{2}f''(x) = 0 \quad (3)$$

Looking for a small h, such that

$$hf'(x) = f(x) + \frac{h^2}{2}f''(x) \quad (4)$$

$$\Rightarrow h = \frac{f(x)}{f'(x)} + \frac{h^2 f''(x)}{2f'(x)} \quad (5)$$

We now use Adomian decomposition method to get

$$h = c + N(h) \quad (6)$$

where $c = \frac{f(x)}{f'(x)}$ and $N(h) = \frac{h^2 f''(x)}{2f'(x)} = \frac{h^2}{2}$ (on the assumption that $\frac{f''(x)}{f'(x)} \approx 1$)

Let $H_m = h_0 + h_1 + \cdots + h_m = h_0 + A_0 + A_1 + \cdots + A_{m-1}$ denote the $(m+1)$terms approximation of *h*.

For $m = 0$

$$H_0 = h_0 = c = \frac{f(x)}{f'(x)}$$

If α is a root then

$$\alpha = x - h \approx x - H_0 = x - \frac{f(x)}{f'(x)}$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

For $m = 1$

$$h \approx H_1 = h_0 + h_1 = h_0 + A_0$$

$$h_0 = c = \frac{f(x)}{f'(x)}$$

$$A_0 = N(h_0) = \frac{h_0^2}{2} = \frac{f^2(x)}{2f'^2(x)}$$

$$\alpha = x - h_1 \approx x - H_1 = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)}{2f'^2(x)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)}{2f'^2(x_n)}$$

For $m = 2$

$$h \approx H_2 = h_0 + h_1 + h_2 = h_0 + A_0 + A_1$$

$$h_0 = c = \frac{f(x)}{f'(x)}\ , h_1 = \frac{f^2(x_n)}{2f'^2(x_n)}$$

$$N'(h) = \frac{d}{dh} N(h) = h$$

$$A_1 = N'(h_0)h_1 = \frac{f(x)}{f'(x)} \times \frac{f^2(x)}{2f'^2(x)} = \frac{f^3(x)}{2f'^3(x)}$$

$$\alpha = x - h_2 \approx x - H_2 = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)}{2f'^2(x)} - \frac{f^3(x)}{2f'^3(x)}$$

The new iterative scheme 1 is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2}{2[f'(x_n)]^2} - \frac{[f(x_n)]^3}{2[f'(x_n)]^3} \qquad (7)$$

3.1.1Convergence analysis

Consider the iteration function g as

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2}{2[f'(x)]^2} - \frac{[f(x)]^3}{2[f'(x)]^3}$$

The following theorem for convergence of the scheme holds.

3.1.2Theorem: Let x^* be the solution of the equation $f(x) = 0,\ f \in C^2$. If $f'(x^*) \neq 0$ then there exists an interval I containing x^* such that for $x_0 \in I$, the iterative scheme (7) converges to the only solution of $f(x) = 0$ belonging to I.

Proof: The statement $f(x^*) = 0,\ f'(x^*) \neq 0 \Rightarrow [f'(x)^*]^2 = 0, [f'(x)^*]^3 \neq 0$ coupled with the fact that $f \in C^2$ implies that $g(x^*) = x^*$ and g continuous differentiable at $x = x^*$.

Thus $|g'(x^*)| = 0 < 1$(Appendix C).There existsan $\varepsilon > 0$ such that for $x \in (x^* - \varepsilon, x^* + \varepsilon)$, $|g'(x)| < 1$, and by the fixed point theorem[7],the iterative scheme $x_{n+1} = g(x)$ converges to the unique solution in that interval.

3.1.3Definition:[7]let $x^* - x_n$ be the truncation error in the *nth*iterate. If there exists a number $p \geq 1$ and a constant $c \neq 0$ such that

$$\lim_{n \to \infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^p} = c\ ,$$

then p is called the order of convergence of the method.

Hence, the following theorem holds:

3.1.4Theorem: consider the nonlinear equation $f(x)=0.$ suppose $f\in C^4.$ Then for the iterative method (7), the convergence is at least of order 3.

Proof;

Consider the iterative method (7)

$$g(x)=x-\frac{f(x)}{f'(x)}-\frac{[f(x)]^2}{2[f'(x)]^2}-\frac{[f(x)]^3}{2[f'(x)]^3}.$$

$$g'(x^*)=g''(x^*)=0 \text{ (Appendix C)} \tag{8}$$

and

$$g'''(x^*)=2\left\{\frac{f'''(x)}{f'(x)}\right\}-3 \text{ (Appendix C)} \tag{9}$$

We used maple17 software, to find g', g'' and g'''.

From Taylor limited expansion of $g(x_n)$ around x^*, we get, for min $(x_n, x^*)<\xi_n<\max(x_n, x^*)$

$$x_{n+1}-x^*=g(x_n)-g(x^*)$$

$$=g'(x^*)(x_n-x^*)+\frac{g''(x^*)}{2}(x_n-x^*)^2+\frac{g'''(\xi_n)}{6}(x_n-x^*)^3 \tag{10}$$

From equations(8)and (10)and for $x_n\neq x^*$, we get

$$\frac{(x_{n+1}-x^*)}{(x_n-x^*)^3}=\frac{g'''(\xi_n)}{6}$$

The statement $f\in C^4$ implies that $g\in C^3$ in the neighborhood of interest of $x=x^*$. Hence for $g'''(x^*)\neq 0$

$$\lim_{n\to\infty}\frac{(x_{n+1}-x^*)}{(x_n-x^*)^3}=\frac{g'''\left(\lim_{n\to\infty}\xi_n\right)}{6}=\frac{g'''(x^*)}{6}\neq 0.$$

$\Rightarrow$ The new scheme 1 is of order three.

3.2*New iterative scheme 2*

Consider the Taylor's expansion for higher order around x, with $x-x_0=h$, to obtain

$$f(x-h)=f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+\cdots$$

Truncating for higher order values of the series from the fourth order and equating to zero, we obtain

$$f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(x)=0 \tag{11}$$

Looking for a small h,

$$hf'(x)=f(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(x) \tag{12}$$

$$\Rightarrow h=\frac{f(x)}{f'(x)}+\frac{h^2 f''(x)}{2f'(x)}-\frac{h^3 f'''(x)}{6f'(x)} \tag{13}$$

Applying Adomian decomposition method, we get

$h=c+N(h)$, where $c=\frac{f(x)}{f'(x)}$ and $N(h)=\frac{h^2}{2}-\frac{h^3 f'''(x)}{6f'(x)}$, on the assumption that $\frac{f''(x)}{f'(x)}\approx 1$

For $m=0$

$$H_0=h_0=c=\frac{f(x)}{f'(x)}$$

If α is the root of the equation then

$$\alpha = x - h \approx x - H_0 = x - \frac{f(x)}{f'(x)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x\ )}$$

For $m = 1$

$$h \approx H_1 = h_0 + h_1 = h_0 + A_o$$

$$A_0 = N(h_0) = \frac{h^2}{2} - \frac{h^3 f'''(x)}{6f'(x)} = \frac{f^2(x)}{2f'^2(x)} - \frac{f^3(x)f'''(x)}{6f'^4(x)}$$

$$h \approx H_1 = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2}{2[f'(x)]^2} - \frac{[f(x)]^3 f'''(x)}{6[f'(x)]^4}$$

The new iterative scheme 2 is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2}{2[f'(x_n)]^2} - \frac{[f(x_n)]^3 f'''(x_n)}{6[f'(x_n)]^4} \qquad (14)$$

3.2.1Convergence analysis

Consider the iteration function g as

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2}{2[f'(x)]^2} - \frac{[f(x)]^3 f'''(x_n)}{6[f'(x)]^4}.$$

The following theorem for convergence of the scheme holds.

3.2.2Theorem: Let x^* be the solution of the equation $f(x) = 0,\ f \in C^4$. If $f'(x^*) \neq 0$ then there exists an interval I containing x^* such that for $x_0 \in I$, the iterative scheme (14) converges to the only solution of $f(x) = 0$ belonging to I.

Proof: The statement $f(x^*) = 0,\ f'(x^*) \neq 0 \Rightarrow [f'(x)^*]^2 = 0, [f'(x)^*]^4 \neq 0$ coupled with the fact that $f \in C^4$ implies that $g(x^*) = x^*$ and g continuous differentiable at $x = x^*$.

Thus $|g'(x^*)| = 0 < 1$(Appendix D). There exists an $\varepsilon > 0$ such that for $x \in (x^* - \varepsilon, x^* + \varepsilon)$, $|g'(x)| < 1$, and by the fixed point theorem[7],the iterative scheme $x_{n+1} = g(x)$ converges to the unique solution in that interval.

The following theorem on the order of convergence holds.

3.2.3Theorem: consider the nonlinear equation $f(x) = 0$. suppose $f \in C^6$. Then for the iterative method (14), the convergence is at least of order 3.

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2}{2[f'(x)]^2} - \frac{[f(x)]^3 f'''(x_n)}{6[f'(x)]^4}.$$

$$g'(x^*) = g''(x^*) = 0 \text{ (Appendix D)} \qquad (15)$$

and

$$g'''(x^*) = \frac{f'''(x^*)}{f'(x^*)} \text{(Appendix D)(16)}$$

We used maple17 software, to find g', g'' and g'''.

From Taylor limited expansion of $g(x_n)$ around x^*, we get, for $\min(x_n, x^*) < \xi_n < \max(x_n, x^*)$

$$x_{n+1} - x^* = g(x_n) - g(x^*)$$

$$= g'(x^*)(x_n - x^*) + \frac{g''(x^*)}{2}(x_n - x^*)^2 + \frac{g'''(\xi_n)}{6}(x_n - x^*)^3 \quad (17)$$

From equation (15)and (17) and for $x_n \neq x^*$, we get

$$\frac{x_{n+1} - x^*}{(x_n - x^*)^3} = \frac{g'''(\xi_n)}{6}.$$

The statement $f \in C^4$ implies that $g \in C^6$ in the neighborhood of interest of $x = x^*$. Hence for $g'''(x^*) \neq 0$

$$\lim_{n\to\infty} \frac{x_{n+1} - x^*}{(x_n - x^*)^3} = \frac{g'''\left(\lim_{n\to\infty} \xi_n\right)}{6} = \frac{g'''(x^*)}{6} \neq 0.$$

$\Rightarrow$ The new scheme 2 is of order three.

To demonstrate the performance of the two new schemes, we constructed twenty five examples of different natures as in Appendix A. With those examples, we compared the two methods with the following earlier methods:

i. Adomian decomposition method
ii. Abbasbandy
iii. Basto et al and
iv. Newton-Raphson

The comparison was carried out in terms of the number of iterations obtained using the methods. In each case, the comparison was done only for those methods which converge for the particular numerical example. The results for the comparison are given in Table E under Appendix B.

4.0 Analysis of result

An analysis of variance was carried out on the results obtained in Table E andthefollowing conclusion can be drawn considering the confidence interval of 95%.Since p=0.01<0.05 for the number of iterations, we conclude that there is significant difference between the number of iterations obtained for the different methods (see Table A).On the other hand, since p=0.294>0.05 for the solutions,we conclude that there is no significant difference in the average solutions obtained using individual methods. The actual difference is in the number of iterations which was investigated further, using the Duncan Multiple range (Post Hoc Test) test displayed in Table B.

From the Duncan Multiple range, we can deduce that in the first homogeneous subset, we have Basto et al, Newton Raphson and New Scheme 1 with the least number of iterations. In the second homogeneous subset, we have Abbasbandy and New Scheme 2 with higher number of iterations. In the last homogeneous subset, we have Adomian method with the highest number of iterations. Full details are shown by the descriptive statisticswhich is displayed in Table C.

Two-way ANOVA Results

TableA: Significant difference effects

Source	Sum of Squares	df	Mean Square	F	Sig.
Iteration	58.416	7	8.345	2.994	0.010
Solutions	142.660	44	3.242	1.163	0.294
Iteration * Solutions	62.222	36	1.728	0.620	0.935
Error	156.083	56	2.787		
Total	424.166	14 4			

Table B: Performance of different homogeneous subsets

Method	N	Subset for alpha = 0.05		
		1	2	3
Basto et al	25	2.24		
Newton Raphson	25	2.48		
New Scheme 1	25	2.84		
Abbasbandy	22		2.95	
New Scheme 2	25		3.36	
Adomian	23			3.83
Means for groups in homogeneous subsets are displayed.				

Table C: Descriptive statistics on the Number of Iterations

		Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean		Minimum	Maximum
					Lower Bound	Upper Bound		
Newton Raphson	5	2.48	0.770	0.154	2.16	2.80	1	4
Basto et al	5	2.24	1.268	0.254	1.72	2.76	1	7
Abbasbandy	2	2.95	1.495	0.319	2.29	3.62	1	7
New Scheme 1	5	2.84	1.179	0.236	2.35	3.33	1	7
New Scheme 2	5	3.36	1.753	0.351	2.64	4.08	1	9
Adomian	3	3.83	1.922	0.401	2.99	4.66	1	8

Table C shows that the individual methods have a minimum of one iteration and the new scheme 2 has the highest number of iterations. The Newton-Raphson method displayed itselegance by having a maximum of four iterations, while Basto et al, Abbasbany and the new scheme 1have equal number of highest iterations of seven. This shows that the new scheme 1performs equally and efficiently like the best existing methods.

5.0 Conclusion

We have presented two simple iterative schemes, which can be used to find roots of nonlinear functional equations.The two schemes are derived from Taylor's series expansion and Adomian method,with the assumption that $\frac{f''(x)}{f'(x)} \approx 1$. In the first scheme three terms of the Taylor's series are used while in the second scheme, four terms of series are used. The New schemes have been constructed by also applying Adomian decomposition method to increase accuracy. The schemes are compared with Newton method[1],Basto et al[5], Abbasbandy[4] and Adomian method[2]and from numerical results, it is shown that the schemes are competitive with the other methods.In terms of computational cost, the New schemes are at an advantage since they are free of the second derivative making them much easier and timesaving.From the numerical results, the New scheme 1appears to be more robust than the New scheme 2.

APPENDIX A

1. $e^{2x}+2x+0.1=0$
2. $2x^3-x^2-7x+6=0$
3. $x^3-9x+1=0$
4. $x^3-3x-1.06=0$
5. $x^3-6x+4=0$
6. $2x-3\sin x-5=0$
7. $x^3-3x+1=0$
8. $3x-\ln x-16=0$
9. $\cos x-2x+3=0$
10. $x+\ln x-2=0$
11. $x^4-12x+7=0$
12. $x^3-x-0.1=0$
13. $x+\sin x+0.5=0$
14. $x-2-e^{-x}=0$
15. $x^3+4x^2+8x+8=0$
16. $3x-\cos x-1=0$
17. $2x-\ln x-7=0$
18. $x^2-1.25x+0.25=0$
19. $x^2+5.15x+2.37=0$
20. $x^4-11x+8=0$
21. $x^3-4x+2=0$
22. $\ln x-x+3=0$
23. $e^x-x-3=0$
24. $2x^2-12x+11=0$
25. $x^3-4x-1=0$

APPENDIX B

Table E: Comparison Between Number Of Iterations For Twenty Five Different Examples.

Problems	Initial starting point, x_0	Methods used/ Number of Iterations						
		NR	BS	ABB	NS1	NS2	AD	
1	-0.05	3	2	5	2	2	NC	
2	0.86	4	2	NC	3	4	4	
3	0.11	1	2	2	1	2	6	
4	-0.35	2	1	2	2	2	8	
5	0.67	3	2	5	3	3	3	
6	2.5	3	2	NC	3	4	7	
7	0.33	3	2	3	2	2	3	
8	5.3	2	2	3	4	4	1	
9	1.5	2	2	2	2	2	1	
10	2	3	2	1	4	4	NC	
11	0.58	2	4	7	2	3	4	
12	-0.1	2	1	1	1	1	2	
13	-0.5	3	2	3	3	3	5	
14	2	3	2	2	3	3	6	
15	-1	1	4	4	7	6	6	
16	0.33	3	7	3	3	3	3	
17	3.5	3	2	2	3	3	2	

18	0.2	3	2	2	2	3	4	
19	-0.2	2	1	5	4	3	2	
20	0.73	3	3	4	3	3	2	
21	0.5	2	2	2	3	3	6	
22	3	3	3	NC	3	7	3	
23	-3	2	1	2	3	2	4	
24	0.92	3	2	3	3	9	3	
25	-0.25	1	1	2	2	4	3	

Key:
NR=Newton-Raphson, BS=Basto et al, ABB=Abbasbandy, NS1=New scheme 1, NS2=New scheme 2, AD=Adomian.
NC=Not converging.

APPENDIX C

$$g(x) = x - \frac{\{f(x)\}}{\{f'(x)\}} - \frac{1}{2}\frac{\{f(x)\}^2}{\{f'(x)\}^2} - \frac{1}{2}\frac{\{f(x)\}^3}{\{f'(x)\}^3}$$

$$g'(x) = \frac{\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} - \frac{\{f(x)\}}{\left\{\frac{d}{dx}f(x)\right\}} + \frac{\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{3}{2}\frac{\{f(x)\}^2}{\left\{\frac{d}{dx}f(x)\right\}^2} + \frac{3}{2}\frac{\{f(x)\}^3\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4}$$

$$\Rightarrow g'(x) = 0$$

$$g''(x) = \frac{\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}} - \frac{2\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^3} + \frac{\{f(x)\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} - 1 + \frac{3\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} - \frac{3\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{3\{f(x)\}}{\left\{\frac{d}{dx}f(x)\right\}} + \frac{15}{2}\frac{\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{6\{f(x)\}^3\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^5} + \frac{3}{2}\frac{\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4}$$

$$\Rightarrow g''(x) = \frac{f''(x)}{f'(x)} - 1 = 0 \text{ since } \frac{f''(x)}{f'(x)} \approx 1$$

$\therefore g''(x)=0$

$g'''(x)=$

$$-3-\frac{12\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^3}-\frac{81}{2}\frac{\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^4}+\frac{6\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^4}$$

$$-\frac{6\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}+\frac{12\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^5}$$

$$-\frac{9\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4}+\frac{30\{f(x)\}^3\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^6}$$

$$-\frac{18\{f(x)\}^3\left\{\frac{d^2}{dx^2}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5}+\frac{\{f(x)\}^2\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}$$

$$+\frac{3}{2}\frac{\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4}-\frac{3\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^2}+\frac{\{f(x)\}\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2}$$

$$+\frac{2\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}}+\frac{5\{f(x)\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2}+\frac{12\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}$$

$$+\frac{3\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}}+\frac{18\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2}$$

$$\Rightarrow g'''(x)=-3-3\left\{\frac{f''(x)}{f'(x)}\right\}^2+2\left\{\frac{f'''(x)}{f'(x)}\right\}+3\left\{\frac{f''(x)}{f'(x)}\right\}=2\left\{\frac{f'''(x)}{f'(x)}\right\}-3$$

$$\therefore g'''(x)=2\left\{\frac{f'''(x)}{f'(x)}\right\}-3$$

APPENDIX D

$$g(x)=x-\frac{\{f(x)\}}{\{f'(x)\}}-\frac{1}{2}\frac{\{f(x)\}^2}{\{f'(x)\}^2}-\frac{1}{6}\frac{\{f(x)\}^3\cdot\{f'''(x)\}}{\{f'(x)\}^4}$$

$$g'(x) = \frac{\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} - \frac{\{f(x)\}}{\left\{\frac{d}{dx}f(x)\right\}} + \frac{\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{1}{2}\frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{1}{6}\frac{\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{2}{3}\frac{\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5}$$

$$\Rightarrow g'(x) = 0$$

$$g''(x) = \frac{\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}} - \frac{2\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^3} - 1 + \frac{3\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} - \frac{3\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} + \frac{7}{2}\frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} - \frac{\{f(x)\}^2\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{1}{6}\frac{\{f(x)\}^3\left\{\frac{d^5}{dx^5}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{4}{3}\frac{\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5} - \frac{10}{3}\frac{\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^6} + \frac{2}{3}\frac{\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^5}$$

$$\Rightarrow g''(x) = \frac{f''(x)}{f'(x)} - 1 = 0 \text{ since } \frac{f''(x)}{f'(x)} \approx 1$$

$$\therefore g''(x) = 0$$

$$g'''(x) =$$

$$-\frac{12\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{3\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^2} - \frac{3}{2}\frac{\{f(x)\}^2\left\{\frac{d^5}{dx^5}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}$$

$$-\frac{1}{6}\frac{\{f(x)\}^3\left\{\frac{d^6}{dx^6}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}} + \frac{\{f(x)\}^2\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}$$

$$+\frac{6\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{3\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}$$

$$+\frac{12\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^5} + \frac{21}{2}\frac{\{f(x)\}^2\left\{\frac{d^4}{dx^4}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4}$$

$$+\frac{11}{2}\frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^4} - \frac{2\{f(x)\}\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2}$$

$$+\frac{2\{f(x)\}^3\left\{\frac{d^5}{dx^5}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5} + \frac{8}{3}\frac{\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5}$$

$$-\frac{9\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{5\{f(x)\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} + \frac{3\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}}$$

$$-\frac{24\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^5} - \frac{10\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^6}$$

$$+\frac{20\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^7} - \frac{10\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^6}$$

$$\Rightarrow g'''(x) = -3\left\{\frac{f''(x)}{f'(x)}\right\}^2 + \frac{f'''(x)}{f'(x)} + 3\frac{f''(x)}{f'(x)} = \frac{f'''(x)}{f'(x)}$$

$$\therefore g'''(x) = \frac{f'''(x)}{f'(x)}$$

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