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A Class of Block Hybrid Methods with Continuous Coefficients for Stiff Initial Value Problems in Ordinary Differential Equations

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Abstract

A class of block hybrid methods (BHMs) based on the Generalized Adams methods for step numbers three and five with one off step point each is presented in this paper for the solution of stiff ordinary differential equations (ODEs). The construction of these block methods is based on the approach of collocation and interpolation. These methods are A-stable, a basic requirement of linear multistep methods for the solution of stiff problems. Numerical results of experiments conducted using the BHMs and their conventional counterpart of the same step size reveals the superiority of the BHM over their conventional ones especially in the case of the three step BHM.

***Keywords*:** Initial value problems, Ordinary Differential equations, Block Hybrid methods

1.0 Introduction

In this paper, we shall consider the numerical solution of the stiff initial value problem (IVP)

$$y' = f(x, y), \quad y(x_0) = y_0, \quad a \le x \le b \tag{1}$$

In recent years, researchers have focused on the development of efficient algorithms for the numerical solution of (1) since most of the problems we encounter in real life are stiff in nature. The algorithm developed by Gear [1] for the solution of (1) remains one of the most efficient general purpose algorithms because of the relative ease with which order and step size may be changed and the possibility of using high order highly stable schemes [see Cash [2]].Recently, a class of continuous Adams formulas including their hybrids for solving (1) have been constructed based on the multistep collocation approach by Onumanyi et.al. [3,4].

Furthermore, regarding the solution of (1), Ibrahim et.al [5] has proposed block backward differentiation formulae (BDF) using Lagrange interpolation with two back values to improve on the well-known BDF. This approach reduces the computational effort involve when using the conventional BDF but unfortunately accuracy is not guaranteed.

Brugnano and Trigiante [6] modified the Adams methods into Generalized Adams methods (GAMs) which are A-stable and have good stability properties for very high step numbers suitable for numerically solving (1). These methods were used as boundary value methods (BVMs) so as to overcome the problem posed by the Dahlquist barrier theorem and the impossibility to define stable, high order symplectic methods. They gave the generalization of the methods as BVMs by the formula

$$y_{n+v} - y_{n+v-1} = h\sum_{i=0}^{k} \beta_i f_{n+i} \tag{2}$$

where $v = \begin{cases} \frac{k+1}{2}, & \text{for odd k} \\ \frac{k}{2}, & \text{for even k} \end{cases}$

Kumleng [7] extended the idea of Brugnano and Trigiante [6] by obtaining both the block Generalized Adams methods (BGAMs) and block hybrid Generalized Adams methods (BHGAMs) for step numbers $k = 2,3,...,10.$ The block methods in

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Kumleng [7] are mostly A –stable and are applied in block form for the solution of system of stiff ODEs. This approach eliminates the need for starting values as the discrete schemes obtained are simultaneously used.

Kumleng et al [8] has also considered the formulation of a new class of block methods similar to the GAMs of Brugnano and Trigiante [6] for the odd step numbers only (k =3 and 5). The difference between this new class and the GAMs is in the choice of ν in (2), ν is chosen as $\frac{k+1}{2}$ for the GAMs in [6]while in Kumleng et al [8], it is chosen as $\frac{k-1}{2}$. This class possesses most of the good stability properties of the GAMs in [6] making them suitable for solving stiff systems.

In this paper, we broaden the idea of Kumleng *et.al* [8] by constructing the hybrid forms of their methods, this is because of the advantages that the hybrid methods have over their conventional counterparts. This we do by incorporating an offstep point in the last subinterval $[x_{n+k-1}, x_{n+k}]$ for each of the methods. This approach can increase the accuracy of the methods for the solution of (1).

2.0 The New Hybrid Block Methods (BHMs)

In this section, the three and five steps BHMs based on the GAMs of Brugnano and Trigiante [6] shall be constructed using the continuous finite difference approximation and the interpolation and collocation criteria described by Lie and Norsett [9] and the block multistep methods of Onumanyi et.al [3].

We defined the continuous form of the k-step new hybrid method based on interpolation and collocation method as

$$y_{n+\nu} - y_{n+\nu-1} = h\sum_{i=0}^{k}\beta_i(x)f_{n+i} + h\beta_\mu(x)f_{n+\mu} \qquad (3)$$

where $\nu = \frac{k-1}{2}$, $\mu = \frac{5}{2}, \frac{9}{2}, \ldots$ and $\beta_i(x)$ and $\beta_\mu(x)$ are the continuous coefficients of the method. The work in this paper is restricted to case k = 3 and 5.

2.1 Derivation of the New Block Hybrid Methods (BHMs)

This section is concerned with the construction of the proposed BHMs for the two conventional methods of kumleng*et.al* [8] incorporating one off step point in each case. This technique is based on the continuous finite difference approximation approach using the interpolation and collocation criteria of Lie and Norsett [9] called multistep collocation and block multistep method by Onumanyi [3].

2.2 Construction of BHM k=3, $\mu = \frac{5}{2}$

The BHM for k=3 has its continuous form based on (3) as

$$y(x) = \alpha_0(x)y_n + h\sum_{j=0}^{3}\beta_j(x)f_{n+j} + h\beta_\mu(x)f_{n+\mu},\ j = 0,1,2,3,\ \mu = \tfrac{5}{2} \qquad (4)$$

Using the approach in Onumanyi [3], a matrix D for the method (4) is arrived at as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \\ 0 & 1 & 2x_{n+\frac{5}{2}} & 3x_{n+\frac{5}{2}}^2 & 4x_{n+\frac{5}{2}}^3 & 5x_{n+\frac{5}{2}}^4 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 \end{bmatrix} \qquad (5)$$

From the inverse of (5), the continuous coefficients of the method are obtained as

$$\begin{aligned} &\alpha_1(x) = 1 \\ &\beta_0(x) = \lambda - \frac{67\lambda^2}{60h} + \frac{26\lambda^3}{45h^2} - \frac{17\lambda^4}{120h^3} + \frac{\lambda^5}{75h^4},\ \beta_1(x) = \frac{5\lambda^2}{2h} - \frac{37\lambda^3}{18h^2} + \frac{5\lambda^4}{8h^3} - \frac{\lambda^5}{15h^4} \\ &\beta_2(x) = -\frac{15\lambda^2}{4h} + \frac{13\lambda^3}{3h^2} - \frac{13\lambda^4}{8h^3} + \frac{\lambda^5}{5h^4},\ \beta_{\frac{5}{2}}(x) = \frac{16\lambda^2}{5h} - \frac{176\lambda^3}{45h^2} + \frac{8\lambda^4}{5h^3} - \frac{16\lambda^5}{75h^4} \\ &\beta_3(x) = -\frac{5\lambda^2}{6h} + \frac{19\lambda^3}{18h^2} - \frac{11\lambda^4}{24h^3} + \frac{\lambda^5}{15h^4} \end{aligned} \qquad (6)$$

Substituting the continuous coefficients (6) into (4) gives the continuous interpolant of the method as

$$y(\lambda + x_n) = y_n + \left(\lambda - \frac{67\lambda^2}{60h} + \frac{26\lambda^3}{45h^2} - \frac{17\lambda^4}{120h^3} + \frac{\lambda^5}{75h^4}\right) f_n + \left(\frac{5\lambda^2}{2h} - \frac{37\lambda^3}{18h^2} + \frac{5\lambda^4}{8h^3} - \frac{\lambda^5}{15h^4}\right) f_{n+1} + \left(-\frac{15\lambda^2}{4h} + \frac{13\lambda^3}{3h^2} - \frac{13\lambda^4}{8h^3} + \frac{\lambda^5}{5h^4}\right) f_{n+2} + \left(\frac{16\lambda^2}{5h} - \frac{176\lambda^3}{45h^2} + \frac{8\lambda^4}{5h^3} - \frac{16\lambda^5}{75h^4}\right) f_{n+\frac{5}{2}} + \left(-\frac{5\lambda^2}{6h} + \frac{19\lambda^3}{18h^2} - \frac{11\lambda^4}{24h^3} + \frac{\lambda^5}{15h^4}\right) f_{n+3} \quad (7)$$

where $\lambda \in [0, 3h] = x - x_n$. The continuous interpolant (7) evaluated at $\lambda = h,\ 2h,\ \frac{5}{2}h,\ 3h$ produces the BHM method for k=3 with $\mu = \frac{5}{2}$ used in block form for the integration of ordinary differential equations

$$\begin{aligned} y_{n+1} - y_n &= \frac{h}{1800}(599 f_n + 1805 f_{n+1} - 1515 f_{n+2} + 1216 f_{n+\frac{5}{2}} - 305 f_{n+3}) \\ y_{n+2} - y_n &= \frac{h}{225}(71 f_n + 320 f_{n+1} + 15 f_{n+2} + 64 f_{n+\frac{5}{2}} - 20 f_{n+3}) \\ y_{n+\frac{5}{2}} - y_n &= \frac{h}{1152}(365 f_n + 1625 f_{n+1} + 375 f_{n+2} + 640 f_{n+\frac{5}{2}} - 125 f_{n+3}) \\ y_{n+3} - y_n &= \frac{h}{200}(63 f_n + 285 f_{n+1} + 45 f_{n+2} + 192 f_{n+\frac{5}{2}} + 15 f_{n+3}) \end{aligned} \quad (8)$$

2.3 Construction of BHM k=5, $\mu = \frac{9}{2}$

The BHM for k=5 has its continuous form based on (3) as

$$y(x) = \alpha_1(x) y_{n+1} + h\sum_{j=0}^{5} \beta_j(x) f_{n+j} + h\beta_\mu(x) f_{n+\mu},\ j = 0,1,2,3,4,5,\ \mu = \tfrac{9}{2} \quad (9)$$

Using similar approach as in the case k=3 above gives the matrix D for the method (9) as

$$D = \begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & 7x_{n+2}^6 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & 7x_{n+3}^6 \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 \\ 0 & 1 & 2x_{n+\frac{9}{2}} & 3x_{n+\frac{9}{2}}^2 & 4x_{n+\frac{9}{2}}^3 & 5x_{n+\frac{9}{2}}^4 & 6x_{n+\frac{9}{2}}^5 & 7x_{n+\frac{9}{2}}^6 \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 \end{bmatrix} \quad (10)$$

and the continuous coefficients of the method are obtained as

$$\begin{aligned} \alpha_1(x) &= y_{n+1} \\ \beta_0(x) &= -\tfrac{28199h}{90720} + \lambda - \tfrac{451\lambda^2}{360h} + \tfrac{2573\lambda^3}{3240h^2} - \tfrac{9\lambda^4}{32h^3} + \tfrac{61\lambda^5}{1080h^4} - \tfrac{13\lambda^6}{2160h^5} + \tfrac{\lambda^7}{3780h^6} \\ \beta_1(x) &= -\tfrac{78553h}{70560} + \tfrac{45\lambda^2}{14h} - \tfrac{271\lambda^3}{84h^2} + \tfrac{947\lambda^4}{672h^3} - \tfrac{67\lambda^5}{210h^4} + \tfrac{37\lambda^6}{1008h^5} - \tfrac{\lambda^7}{588h^6} \\ \beta_2(x) &= \tfrac{4519h}{5040} - \tfrac{9\lambda^2}{2h} + \tfrac{361\lambda^3}{60h^2} - \tfrac{149\lambda^4}{48h^3} + \tfrac{47\lambda^5}{60h^4} - \tfrac{7\lambda^6}{72h^5} + \tfrac{\lambda^7}{210h^6} \\ \beta_3(x) &= -\tfrac{13691h}{15120} + \tfrac{5\lambda^2}{h} - \tfrac{391\lambda^3}{54h^2} + \tfrac{199\lambda^4}{48h^3} - \tfrac{103\lambda^5}{90h^4} + \tfrac{11\lambda^6}{72h^5} - \tfrac{\lambda^7}{126h^6} \\ \beta_4(x) &= \tfrac{9841h}{10080} - \tfrac{45\lambda^2}{8h} + \tfrac{203\lambda^3}{24h^2} - \tfrac{491\lambda^4}{96h^3} + \tfrac{181\lambda^5}{120h^4} - \tfrac{31\lambda^6}{144h^5} + \tfrac{\lambda^7}{84h^6} \\ \beta_{\frac{9}{2}}(x) &= -\tfrac{13808h}{19845} + \tfrac{256\lambda^2}{63h} - \tfrac{17536\lambda^3}{2835h^2} + \tfrac{80\lambda^4}{21h^3} - \tfrac{1088\lambda^5}{945h^4} + \tfrac{32\lambda^6}{189h^5} - \tfrac{64\lambda^7}{6615h^6} \\ \beta_5(x) &= \tfrac{1537h}{10080} - \tfrac{9\lambda^2}{10h} + \tfrac{83\lambda^3}{60h^2} - \tfrac{83\lambda^4}{96h^3} + \tfrac{4\lambda^5}{15h^4} - \tfrac{29\lambda^6}{720h^5} + \tfrac{\lambda^7}{420h^6} \end{aligned} \quad (11)$$

Substituting the continuous coefficients (11) into (9) gives the continuous interpolant of the method as

$$
\begin{aligned}
y(\lambda + x_n) = {} & y_{n+1} + \left(-\tfrac{28199h}{90720} + \lambda - \tfrac{451\lambda^2}{360h} + \tfrac{2573\lambda^3}{3240h^2} - \tfrac{9\lambda^4}{32h^3} + \tfrac{61\lambda^5}{1080h^4} - \tfrac{13\lambda^6}{2160h^5} + \tfrac{\lambda^7}{3780h^6}\right) f_n + \\
& \left(-\tfrac{78553h}{70560} + \tfrac{45\lambda^2}{14h} - \tfrac{271\lambda^3}{84h^2} + \tfrac{947\lambda^4}{672h^3} - \tfrac{67\lambda^5}{210h^4} + \tfrac{37\lambda^6}{1008h^5} - \tfrac{\lambda^7}{588h^6}\right) f_{n+1} + \\
& \left(\tfrac{4519h}{5040} - \tfrac{9\lambda^2}{2h} + \tfrac{361\lambda^3}{60h^2} - \tfrac{149\lambda^4}{48h^3} + \tfrac{47\lambda^5}{60h^4} - \tfrac{7\lambda^6}{72h^5} + \tfrac{\lambda^7}{210h^6}\right) f_{n+2} + \\
& \left(-\tfrac{13691h}{15120} + \tfrac{5\lambda^2}{h} - \tfrac{391\lambda^3}{54h^2} + \tfrac{199\lambda^4}{48h^3} - \tfrac{103\lambda^5}{90h^4} + \tfrac{11\lambda^6}{72h^5} - \tfrac{\lambda^7}{126h^6}\right) f_{n+3} \\
& \left(\tfrac{9841h}{10080} - \tfrac{45\lambda^2}{8h} + \tfrac{203\lambda^3}{24h^2} - \tfrac{491\lambda^4}{96h^3} + \tfrac{181\lambda^5}{120h^4} - \tfrac{31\lambda^6}{144h^5} + \tfrac{\lambda^7}{84h^6}\right) f_{n+4} \\
& \left(-\tfrac{13808h}{19845} + \tfrac{256\lambda^2}{63h} - \tfrac{17536\lambda^3}{2835h^2} + \tfrac{80\lambda^4}{21h^3} - \tfrac{1088\lambda^5}{945h^4} + \tfrac{32\lambda^6}{189h^5} - \tfrac{64\lambda^7}{6615h^6}\right) f_{n+\frac{9}{2}} \\
& \left(\tfrac{1537h}{10080} - \tfrac{9\lambda^2}{10h} + \tfrac{83\lambda^3}{60h^2} - \tfrac{83\lambda^4}{96h^3} + \tfrac{4\lambda^5}{15h^4} - \tfrac{29\lambda^6}{720h^5} + \tfrac{\lambda^7}{420h^6}\right) f_{n+5}
\end{aligned} \tag{12}
$$

where $\lambda \in [0, 5h] = x - x_n$. The continuous interpolant (12) evaluated at $\lambda = 0, 2h,\ 3h, 4h, \frac{9}{2}h,\ 5h$ produces the BHM method for k=5 with $\mu = \frac{9}{2}$ used in block form for the integration of ordinary differential equations

$$
\begin{aligned}
y_{n+1} - y_n &= \tfrac{h}{635040}(197393 f_n + 706977 f_{n+1} - 569394 f_{n+2} + 575022 f_{n+3} - 619983 f_{n+4} + 441856 f_{n+\frac{9}{2}} - 96831 f_{n+5}) \\
y_{n+2} - y_{n+1} &= \tfrac{h}{635040}(-8113 f_n + 256527 f_{n+1} + 518994 f_{n+2} - 227598 f_{n+3} + 204687 f_{n+4} - 138752 f_{n+\frac{9}{2}} + 29295 f_{n+5}) \\
y_{n+3} - y_{n+1} &= \tfrac{h}{39690}(-3721 f_n + 14544 f_{n+1} + 51534 f_{n+2} + 12894 f_{n+3} + 2709 f_{n+4} - 2560 f_{n+\frac{9}{2}} + 630 f_{n+5}) \\
y_{n+4} - y_{n+1} &= \tfrac{h}{23520}(-259 f_n + 8973 f_{n+1} + 28854 f_{n+2} + 20118 f_{n+3} + 18333 f_{n+4} - 6656 f_{n+\frac{9}{2}} + 1197 f_{n+5}) \\
y_{n+\frac{9}{2}} - y_{n+1} &= \tfrac{h}{414720}(-4459 f_n + 157311 f_{n+1} + 512442 f_{n+2} + 343686 f_{n+3} + 441441 f_{n+4} - 14336 f_{n+\frac{9}{2}} + 15435 f_{n+5}) \\
y_{n+5} - y_{n+1} &= \tfrac{h}{19845}(-224 f_n + 7614 f_{n+1} + 24192 f_{n+2} + 17304 f_{n+3} + 18144 f_{n+4} + 8192 f_{n+\frac{9}{2}} + 4158 f_{n+5})
\end{aligned} \tag{13}
$$

3.0 Analysis of the New Block Methods

In this section, the convergence analysis of the new methods is considered and their regions of absolute stability are plotted.

3.1 Convergence Analysis

In the spirit of Fatunla [10] and Chollom [11], the block methods are represented in a single block r point multi-step method of the form;

$$
A^{(0)} y_m = \sum_{i=1}^{k} A^{(i)} y_{m+1} + h \sum_{i=1}^{k} B^{(i)} f_m \tag{14}
$$

where is a fixed mesh size within a block, $A^{(i)}, B^{(i)},\ i = 0,1,2,...,k$ are $r \times r$ identity matrix while y_m, y_{m+1}, f_m are vectors of numerical estimates.

Definition 1: Zero stability for Block methods

For n = mr, for some integer $m \geq 0$, the block method is zero stable if the roots $R_j,\ j = 1,2,...,k$ of the first characteristic polynomial $\rho(R)$ given by:

$$
\rho(R) = \det\left[\sum_{i=0}^{k} A^{(i)} R^{i}\right] = 0 \tag{15}
$$

satisfies $|R_j| \leq 1$ and for those roots with $|R_j| = 1$, the multiplicity must not exceed two.

The block method (8) expressed in the form of (14) yields

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix}
+ h
\begin{bmatrix} \frac{1805}{1800} & -\frac{1515}{1800} & \frac{1216}{1800} & -\frac{305}{1800} \\ \frac{320}{225} & \frac{15}{225} & \frac{64}{225} & -\frac{20}{225} \\ \frac{1625}{1152} & \frac{375}{1152} & \frac{640}{1152} & -\frac{125}{1152} \\ \frac{285}{200} & \frac{45}{200} & \frac{192}{200} & \frac{15}{200} \end{bmatrix}
\begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{bmatrix}
+
\begin{bmatrix} 0 & 0 & 0 & \frac{599}{1800} \\ 0 & 0 & 0 & \frac{71}{225} \\ 0 & 0 & 0 & \frac{365}{1152} \\ 0 & 0 & 0 & \frac{63}{200} \end{bmatrix}
\begin{bmatrix} f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} \tag{16}
$$

where

$$A^{(0)}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},A^{(1)}=\begin{bmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{bmatrix},B^{(0)}=\begin{bmatrix}\frac{1805}{1800}&-\frac{1515}{1800}&\frac{1216}{1800}&-\frac{305}{1800}\\\frac{320}{225}&\frac{15}{225}&\frac{64}{225}&-\frac{20}{225}\\\frac{1625}{1152}&\frac{375}{1152}&\frac{640}{1152}&-\frac{125}{1152}\\\frac{285}{200}&\frac{45}{200}&\frac{192}{200}&\frac{15}{200}\end{bmatrix},B^{(1)}=\begin{bmatrix}0&0&0&\frac{599}{1800}\\0&0&0&\frac{71}{225}\\0&0&0&\frac{365}{1152}\\0&0&0&\frac{63}{200}\end{bmatrix}$$

Substituting $A^{(0)}, A^{(1)}$ into the characteristic polynomial (15) gives

$$\rho(R)=\left|R\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right|=R^3(R-1)=0\Rightarrow R_1=1,R_2=R_3=R_4=0$$

The block method (8) by definition 1 is zero stable and is of order $p\geq 1$, thus by Henrici[12], the block method (8) is convergent. Following the same approach, the block method (13) is also convergent.

3.2 Order of the New Block Methods

Following the approach in Chollom [11], the order and error constants of the new methods are determined. The methods have the following orders and error constants:

Method	BHM k=3, $\mu=\frac{5}{2}$	BHM k=3, $\mu=\frac{9}{2}$
Order	5	7
Error Constants	$\frac{13}{1200},\frac{7}{900},\frac{25}{3072},\frac{3}{400}$	$-\frac{1759}{211680},-\frac{137}{70560},-\frac{11}{10584},-\frac{13}{7840},-\frac{6811}{4423680},-\frac{4}{2205}$

3.3 Stability Regions of the New Block Methods

In order to plot the absolute stability regions of the proposed block hybrid methods, the approach in Chollom [13] was used where the block methods are reformulated as the General linear methods of Butchers [14] expressed as:

$$\begin{bmatrix}Y\\y_{n+1}\end{bmatrix}=\begin{bmatrix}A&U\\B&V\end{bmatrix}\begin{bmatrix}hf(Y)\\y_{n-1}\end{bmatrix} \qquad (17)$$

The block methods (8) and (13) are expressed in the form of (17) and the coefficient matrices A,B,U, and V are used with a matlab software to produce the regions of absolute stability of the two block methods as shown in Figure 1.

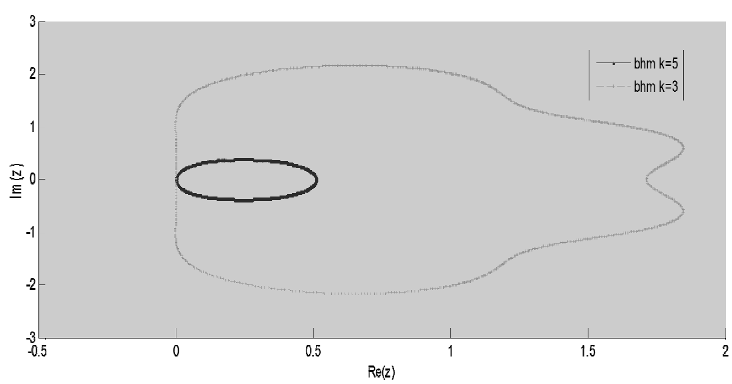

Figure1. Absolute Stability Regions of the Block Hybrid Methods for K=3 and 5

Figure 1 above reveals that the two BHM methods are A-stable since their region of absolute stability contains the whole of the left-hand half plane.

4.0 Numerical Experiments

In this section, we present some numerical results to compare the performance of the new methods with the standard methods for the same step numbers k=3 and k=5. Absolute errors between the exact solution and the proposed block hybrid methods and the methods in Kumleng *et.al* [8] for some selected time steps (2.5, 5.0, 7.5, 10.0) are as shown in Table 1 – 3.

Example 1. Consider the stiff system; see Ali and Hojjati [15]

$$y_1' = -y_1 - 15y_2 + 15e^{-x}, \quad y_1(0) = 1$$
$$y_2' = 15y_1 - y_2 - 15e^{-x}, \quad y_2(0) = 1$$

Its exact solution is $y_1(x) = y_2(x) = e^{-x}$, h=0.01. This system has eigenvalues of large modulus lying close to the imaginary axis $-1 \pm 15i$.

Table 1: Absolute error in example 1

x	yi	Exact solution	Error in BHM k=3	Error in BHM k=5	Error in kumleng et al (2013) k=3	Error in kumleng et al (2013) k=5
2.5	y1	0.082084999	1.22E-15	1.39E-17	3.00E-13	5.55E-17
	y2	0.082084999	1.15E-15	4.16E-17	2.67E-13	8.33E-17
5.0	y1	0.006737947	2.05E-16	8.67E-18	5.30E-14	1.73E-18
	y2	0.006737947	1.59E-16	3.47E-18	3.82E-14	1.91E-17
7.5	y1	0.000553084	2.85E-17	2.49E-18	6.91E-15	9.76E-19
	y2	0.000553084	1.62E-17	3.25E-19	4.00E-15	1.30E-18
10.0	y1	4.53999E-05	3.29E-18	2.10E-19	7.86E-16	9.49E-20
	y2	4.53999E-05	1.46E-18	8.81E-19	3.57E-16	0.00E+00

Example 2. Consider the system of differential equations

$$y_1' = -20y_1 - 0.25y_2 - 19.75y_3, \quad y_1(0) = 1$$
$$y_2' = 20y_1 - 20.25y_2 + 0.25y_3, \quad y_2(0) = 0$$
$$y_3' = 20y_1 - 19.75y_2 - 0.25y_3, \quad y_2(0) = -1$$

The theoretical solution is
$$y_1 = \tfrac{1}{2}[e^{-0.5x} + e^{-20x}(\cos(20x) + \sin(20x))]$$
$$y_2 = \tfrac{1}{2}[e^{-0.5x} - e^{-20x}(\cos(20x) - \sin(20x))]$$
$$y_3 = -\tfrac{1}{2}[e^{-0.5x} + e^{-20x}(\cos(20x) - \sin(20x))]$$
h=0.01, $0 \le x \le 10$.

Table 2: Absolute Error in Example 2.

x	yi	Exact solution	Error in BHM k=3	Error in BHM k=5	Error in kumleng et al (2013) k=3	Error in kumleng et al (2013) k=5
2.5	y1	0.143252398	6.02E-15	2.78E-17	2.94E-12	1.39E-16
	y2	0.143252398	6.05E-15	2.78E-17	2.94E-12	1.39E-16
	y3	-0.143252398	6.05E-15	2.78E-17	2.94E-12	1.39E-16
5.0	y1	0.041042499	3.46E-15	2.78E-17	1.68E-12	0.00
	y2	0.041042499	3.46E-15	2.78E-17	1.68E-12	0.00
	y3	-0.041042499	3.46E-15	2.78E-17	1.68E-12	0.00

7.5	y1	0.011758873	1.48E-15	3.47E-18	7.23E-13	6.94E-18
	y2	0.011758873	1.48E-15	1.74E-18	7.23E-13	5.20E-18
	y3	-0.011758873	1.48E-15	1.74E-18	7.23E-13	5.20E-18
10.0	y1	0.003368973	5.64E-15	4.34E-17	2.76E-13	4.77E-18
	y2	0.003368973	5.65E-15	4.34E-17	2.76E-13	4.77E-18
	y3	-0.003368973	5.65E-15	4.34E-17	2.76E-13	4.77E-18

Example 3. Consider the system

$$y_1' = 998y_1 + 1998y_2, \qquad y_1(0) = 1$$

$$y_2' = -999y_1 - 1999y_2 \qquad y_2(0) = 1$$

The theoretical solution is $y_1(x) = 4e^{-x} - 3e^{-1000x}, y_2(x) = -2e^{-x} + 3e^{-1000x},\ h = 0.01$ and $0 \le x \le 10$

Table3: Absolute Error in Example 3

x	yi	Exact solution	Error in BHM k=3	Error in BHM k=5	Error in kumleng et al (2013) k=3	Error in kumleng et al (2013) k=5
2.5	y1	0.328339994	8.91E-13	5.00E-15	2.14E-10	7.00E-15
	y2	-0.164169997	4.45E-13	2.00E-15	1.07E-10	4.00E-15
5.0	y1	0.026951788	1.46E-13	0.00	3.52E-11	1.00E-15
	y2	-0.013475894	7.30E-14	0.00	1.76E-11	0.00
7.5	y1	0.002212337	1.80E-14	0.00	4.33E-12	0.00
	y2	-0.001106169	9.00E-15	0.00	2.17E-12	0.00
10.0	y1	0.0001816	2.00E-15	0.00	4.74E-13	0.00
	y2	-9.07999E-05	1.00E-15	0.00	2.37E-13	0.00

5.0 Conclusion

Two block hybrid methods with one off step points each for the methods in Kumleng *et.al*[8] have been proposed and implemented as self-starting methods for ordinary differential equations. The good convergent and stability properties of these block hybrid methods make them attractive for the numerical solution of stiff problems. The accuracy of these block methods have been demonstrated on some stiff problems as shown in Table 1 – 3. In all the problems solved, we observed the new block hybrid method [8] converges faster than their conventional block methods in Kumleng *et.al*[8] but for the BHM in (13), we observed that their performance are relatively equal.

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