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A Class of Some Explicit S-Stage of Ordered Runge-Kutta Methods

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Abstract

In this paper, a class of some explicit s-stage of ordered Runge-Kutta methods were investigated. Some explicit schemes were developed based on the first order ordinary differential equation using Taylor series expansion method. These methods were implemented and evaluated on a sampled problem. The error terms from the results show that the methods are accurate, stable and consistence.

***Keywords*:** Runge-Kutta methods, Taylor series.

1.0 Introduction

Linear multistep methods are used for the numerical solution of ordinary differential equations. A numerical method starts from an initial point and then takes a short step forward in time to find the next solution conceptually [1]. The process continues with subsequent steps to map out solution. Single step methods (such as Euler's method) refer only to one previous point and its derivative to determine the current value. Methods such as Runge-Kutta take some intermediate steps (for example, a half step) to obtain a higher order method, but then discard all previous information before taking a second step [2]. Multistep methods attempts to gain efficiency by keeping and using the information from previous steps rather than discarding it. Consequently, multistep methods refer to several previous points and derivative values. In the case of linear multistep methods, a linear combination of the previous points and derivative values are used.

Definition: For a general s-stage explicit Runge-Kutta method, let us consider the initial value problem (IVP) of the form [3]

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{1.1}$$

And let s be an integer (i.e the number of stages) and

$$a_{21}, a_{31}, a_{32}, \ldots\ldots, a_{s1}, a_{s2}, \ldots\ldots, a_{s,s-1}, b_1, b_2, \ldots.., b_s, c_2, \ldots\ldots, c_s$$

be real constants;
then the method

$$K_1 = f(t_0, y_0)$$
$$K_2 = f(t_0 + c_2h, y_o + ha_{21}K_1)$$
$$K_3 = f(t_0 + c_3h, y_0 + h(a_{31}K_1 + a_{32}K_2))$$
$$.$$
$$.$$
$$K_s = f(t_0 + c_sh, y_0 + (a_{s1}K_1 + a_{s2}K_2 + \ldots..a_{s,s-1}K_{s-1}))$$

which can be written as

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$$K_i = f\left(t_{n-1} + c_i h, y_{n-1} + h\sum_{j=1}^{s} a_{ij}K_j\right), 1 \le i \le s$$

$$y_1 = y_0 + h(b_1K_1 + b_2K_2 + ... + b_sK_s$$

Also written as

$$y_n = y_{n-1} + h\sum_{i=1}^{s} b_iK_i$$

Is an s-stage explicit Runge-Kutta method for (1.1). h is a non-negative real constant called the step length of the method. Usually c_i satisfy the conditions

$$c_2 = a_{21}, c_3 = a_{31} + a_{32}, ..., c_s = a_{s1} + a_{s2} + ... + a_{s,s-1}$$

And can generally be written as

$$c_i = \sum_{j=1}^{i-1} a_{ij}$$

Derivation of s-stage explicit Runge-Kutta method

Consider (1.1) and suppose $y = f(t)$ is it solution. Then by Taylor series expansion

$$f(t) = f(0) + \frac{h}{1!}f'(0) + \frac{h^2}{2!}f''(0) + ...$$

Expand $f(t)$ about $t = t_0$

$$f(t) = f(t_0) + \frac{(t-t_0)}{1!}f'(t_0) + \frac{(t-t_0)^2}{2!}f''(t_0) + ...$$

Evaluate at point $t = t_1$

$$f(t_1) = f(t_0) + \frac{(t-t_1)}{1!}f'(t_0) + \frac{(t-t_1)^2}{2!}f''(t_0) + ...$$

Thus a step length $h = t_k - t_{k-1}$

such that $$f(t_1) = f(t_o) + \frac{h}{1!}f'(t_0) + \frac{h^2}{2!}f''(t_0) + ... + \frac{h}{p!}f^p(t_0) + O\left(h^{p+1}\right)$$

So since (1.1) is of order one

$$y_1 = y_0 + hy_0'$$
$$y_2 = y_1 + hy_1'$$
$$y_3 = y_2 + hy_2'$$

Then by successive approximation

$$y_{n+1} = y_n + \frac{h}{1!}y_n' + \frac{h^2}{2!}y_n'' + ... + \frac{h^p}{p!}y_n{}^p + O\left(h^{p+1}\right)$$

And, if $f(t_n, y_n) = f_n$ and so on, then

$$y_{n+1} = y_n + \frac{h}{1!}f_n + \frac{h^2}{2!}\left(\frac{df}{dt}\right)_n + ... + \frac{1}{p!}h^p\left(\frac{dp^{n-1}f}{dt^{n-1}}\right)_n + O\left(h^{p+1}\right)$$

To obtain a general s-stage explicit Runge-Kutta method, we let

$$Y_{n+1} = Y_n + h\phi(t_n, Y_n; h) \qquad (1.2)$$

where $$\phi(t_n, Y_n; h) = \sum_{i=1}^{s} b_iK_i$$

So that (1.2) becomes

$$Y_{n+1} = Y_n + h\sum_{i=1}^{s} b_i K_i$$

With

$$K_i = f\left(t_n + hc_i, Y_n + h\sum_{j=1}^{i-1} a_{ij} K_j\right) \qquad (1.3)$$

Where $c_i = 0$ for an explicit method and (1.3) can be solved for each K_i in turn.

2.0 Order of Runge-Kutta method:

A Runge-Kutta method has order p if for sufficiently smooth problems (1.1), p is the largest integer for which [4]

$$y(t+h) - y(t) - h\phi(t, y(t); h) = O(h^{p+1})$$

And this means that the local truncation error is $O(h^{p+1})$.

3.0 Implementation of the method

Consider the initial value problem [5]

$$\frac{dy}{dt} = 1 + (y-t)^2, \; y(0) = \frac{1}{2} \qquad (1.4)$$

And let

$$y = t + z \qquad (1.5)$$

By means of substituting (1.5) into (1.4), we have

$$\frac{d}{dt}(t+z) = 1 + (t+z-t)^2$$

$$1 + \frac{dz}{dt} = 1 + z^2$$

$$\frac{dz}{dt} = 1 + z^2 - 1$$

$$\frac{dz}{dt} = z^2$$

Separating the variables leads to

$$\frac{dz}{z^2} = dt$$

$$z^{-2} dz = dt$$

On integration

$$\int z^{-2} dz = \int dt$$

$$-z^{-1} = t + C$$ (where C is the constant of integration)

$$-\frac{1}{z} = t + C$$

From (1.5)

$$z = y - t$$

$$-\frac{1}{y-t} = t + C$$

$$-1 = t(y-t) + C(y-t)$$

$$-1 = (t+C)(y-t)$$

$$y - t = \frac{-1}{t + C}$$

$$y(t) = t - \frac{1}{t + C}$$

By the initial condition $y(0) = \frac{1}{2}$

$$y(0) = \frac{1}{2} = 0 - \frac{1}{C}$$

$$\frac{1}{2} = -\frac{1}{C}$$

$$C = -2$$

$$y(t) = t - \frac{1}{t - 2}$$

$$y(t) = t + \frac{1}{2 - t}, t \neq 2$$

Numerical solutions are given to a second order Runge-Kutta method, a third order Runge-Kutta method and a fourth order Runge-Kutta method of the IVP in (1.4), obtaining numerical solutions for values of t up to and including $t = 1$ with a step size of 0.1as found in Table 1.

Table1: Solutions of ordered Runge-Kutta methods with $h = 0.1$

T	True Solution	2nd order Runge-Kutta	3rd order Runge-Kutta	4th order Runge-Kutta
0.0	0.5000	0.5000	0.5000	0.5000
0.1	0.6263	0.5750	0.6263	0.6263
0.2	0.7556	0.6534	0.7556	0.7556
0.3	0.8882	0.7360	0.8882	0.8882
0.4	1.0250	0.8237	1.0250	1.0250
0.5	1.1667	0.9176	1.1667	1.1667
0.6	1.3143	1.0195	1.3143	1.3143
0.7	1.4692	1.1314	1.4692	1.4692
0.8	1.6333	1.2565	1.6333	1.6333
0.9	1.8091	1.3991	1.8091	1.8091
1.0	2.0000	1.5656	1.9999	2.0000

Table 2: Relative Errors of ordered Runge-Kutta methods with $h = 0.1$

T	2nd order Runge-Kutta	3rd order Runge-Kutta	4th order Runge-Kutta
0.0	0.0000	0.0000	0.0000
0.1	8.1910	0.0000	0.0000
0.2	13.5257	0.0000	0.0000
0.3	17.1358	0.0000	0.0000
0.4	19.6390	0.0000	0.0000
0.5	21.3508	0.0000	0.0000
0.6	22.4302	0.0000	0.0000
0.7	22.9921	0.0000	0.0000
0.8	23.0699	0.0000	0.0000
0.9	21.6920	0.0000	0.0000
1.0	21.7200	0.0050	0.0000

4.0 Discussion of Results:

Table 1 gives the solutions of (1.4) for 2nd order, 3rd order and 4th order Runge-Kutta methods to four-decimal-place accuracy. Comparing the solutions of these order methods to the true solutions in the table, it is observed that the values of the 3rd order Runge-Kutta method are much more accurate than the values of the 2nd order Runge-Kutta method and the 4th order Runge-Kutta method gives exact values as the true solution which implies that the 4th order is so accurate. Therefore, 4th order Runge-Kutta method beats the heck out of 2nd order and 3rd order Runge-Kutta methods.

Table 2 gives the values of the relative errors in the 2nd order, 3rd order and 4th order Runge-Kutta methods at $t = 1$ to be 21.72 percent, 0.005 percent and 0 percent respectively. In this comparison, the 3rd order Runge-Kutta method is about 4000 times accurate than the 2nd order Runge-Kutta method and the 4th order Runge-Kutta method is about 4000 and 16000000 times accurate than 3rd order and 2nd order Runge-Kutta methods respectively. The error bounds for each of these methods are given below.

The 2nd order Runge-Kutta method agrees with a Taylor polynomial of degree 2. So, the local truncation error for this method is O$\left(h^3\right)$ and the global truncation error is O$\left(h^2\right)$.

The actual solution is

$$y(t) = t + \frac{1}{(2-t)}, t \neq 2$$

$$y'(t) = 1 + (2-t)^{-2}$$

$$y''(t) = 2(2-t)^{-3}$$

$$y'''(t)\frac{h^3}{3!} = 6(2-t)^{-4}\frac{h^3}{3!} \tag{1.6}$$

with $t = 1$, (1.6) yields a bound of 0.001 on the local truncation error for each of the ten steps when $h = 0.1$

The 3rd order Runge-Kutta method agrees with a Taylor polynomial of degree 3. So, the local truncation error for this method is O$\left(h^3\right)$ Differentiating (1.6) gives

$$y^{(4)}(t)\frac{h^4}{4!} = 24(2-t)^{-5}\frac{h^4}{4!} \tag{1.7}$$

with $t = 1$, (1.7) yields a bound of 0.0001 on the local truncation error for each of the ten steps when $h = 0.1$

The 4th order Runge-Kutta method agrees with a Taylor polynomial of degree 4. So, the local truncation error for this method is O$\left(h^4\right)$ Differentiating (1.7) gives

$$y^{(5)}(t)\frac{h^5}{5!} = 120(2-t)^{-6}\frac{h^5}{5!} \tag{1.8}$$

with $t = 1$, (1.8) yields a bound of 0.00001 on the local truncation error for each of the ten steps when $h = 0.1$

Table 3: Error Bounds of ordered Runge-Kutta methods

Order of Runge-kutta	Error bound
2nd	$y'''(t)\frac{h^3}{3!} = 6(2-t)^{-4}\frac{h^3}{3!}$
3rd	$y^{(4)}(t)\frac{h^4}{4!} = 24(2-t)^{-5}\frac{h^4}{4!}$
4th	$y^{(5)}(t)\frac{h^5}{5!} = 120(2-t)^{-6}\frac{h^5}{5!}$

These show that the Runge-Kutta methods results in a rapid decrease in errors when the step size h is reduced.

5.0 Conclusion:

Investigation carried out on a class of some explicit s-stage of ordered Runge-Kutta methods has shown that the 4^{th} order Runge-Kutta method gives exact values as the true solutions and this implies that the 4^{th} order is so accurate and consistent when compared to the 2^{nd} and 3^{rd} order Runge-Kutta methods.

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