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A Note on Boundary Layer Theory

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Abstract

For the case of laminar boundary layer flow over a flat plate with zero angle of incidence and pressure gradient we derive the relation for energy thickness of Wieghardt [1] (i.e the balance between mechanical energy loss and heat generated by fluid friction), (see also Schlichting [2]. By applying a parabolic velocity profile in the relation for energy thickness and also in Karman-Pohlhausen [3] momentum integral equation an approximate value of the boundary layer thickness is determined. Comparison of the approximate value with the exact Blasius [4] value leads to the determination of the percentage error for this parameter. It is observed that as the boundary layer flow becomes more laminar (i.e $\mathrm{Re} << 5\times10^5$***), where Re is the Reynolds number, the boundary layer thickness increases accordingly with consequent increase in the percentage error.***

Keywords: Laminar, boundary layer, flat plate, zero angle of incidence.

1.0 Introduction

A boundary layer is the layer fluid in the immediate vicinity of a boundary surface where the effects of viscosity are significant. Laminar boundary layer can be loosely classified according to their structure and the circumstances under which they are created. Thus, we have Stokes boundary layer and Blasius boundary layer. The former is the thin layer which develops on an oscillating body, while the later refers to the well-known similarity solution near an attached flat plate held in an oncoming unidirectional flow.

The subject, boundary layer, has been discussed extensively by many authors since the development of the concept by Prandtl [5]. For instance Craft and Lowell [6] applied steady state boundary layer theory to two aspects of oceanic hydrothermal heat flux and in their analysis they showed that, for near-axis model, heat transfer in the hydrothermal boundary layer is greater than the input from steady state generation of the oceanic crust by sea flow spreading.

Habib et al [7] carried out transient calculation of the boundary layer flow over spills using simulation and experimental approaches. They validated their results against experimental data and also made comparison of the simulated results with empirical prediction models.

Dorfman [8] presented a review of universal functions widely used in different areas of boundary layer theory for many years up to the present. In his work he adopted various solutions from many published articles to show the breadth of universal approaches with application in laminar, turbulent and transition boundary layers in solving non-isothermal and conjugate heat transfer problems as well as in planetary boundary layer problems in meteorology.

Other researchers in the subject include, notably Olsson and Turkdogan [9], Mahmoudian and Scales [10], Kim and Changhoon [11], Eyo et al. [12], Huguera [13], Bohr et al. [14], etc.

In this work we incorporate a parabolic velocity profile in the relation for energy thickness and in momentum integral equation to determine the approximate value of the boundary layer thickness. By comparing the approximate value of this parameter with exact value the percentage error of this parameter is also determined. Our analysis is based on plane, incompressible, steady flow and on the assumption of a laminar boundary layer.

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2.0 Balance Between Loss of Mechanical Energy and Heat Generated by Fluid Friction

Let *x*, *y* be the rectangular coordinates with *y* vertically upwards and *u*, *v* the corresponding velocity components, then the equations for laminar boundary layer flow are:

Continuity equation:
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2.1)$$

Bernoulli equation:
$$\frac{p}{\rho}+\frac{1}{2}U^2=const. \qquad (2.2)$$

Navier-Stokes equation:
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \qquad (2.3)$$

where U = free stream velocity, ρ = fluid density, p = fluid pressure,
μ = dynamic viscosity.

2.1 Deduction from Boundary Layer Equations

We shall restrict our analysis to steady state flow. Here $\frac{\partial u}{\partial t}=0$ and $\frac{\partial p}{\partial x}=\frac{dp}{dx}$, so that (2.3) reduces to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\upsilon\frac{\partial^2 u}{\partial y^2} \qquad (2.4)$$

where $v=\frac{\mu}{\rho}$ is the kinematic viscosity.

From Bernoulli's equation(2.2), we find

$$\frac{dp}{dx}=-\rho U\frac{dU}{dx} \qquad (2.5)$$

so that (2.4) becomes

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\upsilon\frac{\partial^2 u}{dy^2} \qquad (2.6)$$

2.2 Analysis of the Work

We multiply each term of eqn. (2.6) by the velocity *u* to get

$$u^2\frac{\partial u}{\partial x}+uv\frac{\partial u}{\partial y}=uU\frac{dU}{dx}+u\upsilon\frac{\partial^2 u}{dy^2} \qquad (2.7)$$

From continuity equation (2.1) we get, by integration, the velocity

$$v=-\int_0^y \frac{\partial u}{\partial x}dy \qquad (2.8)$$

Using (2.8) in (2.7) and integrating the result from $y=0$ to $y=\delta$ wrt *y*, gives

$$\int_0^\delta \left[u^2\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}\left(\int_0^y \frac{\partial u}{\partial x}dy\right)-uU\frac{dU}{dx}\right]dy=\upsilon\int_0^\delta u\frac{\partial^2 u}{dy^2}dy \qquad (2.9)$$

We now simplify each term in the square bracket of (2.9) as follows:

$$\int_0^\delta u^2\frac{\partial u}{\partial x}dy=\frac{1}{3}\int_0^\delta \frac{\partial\left(u^3\right)}{\partial x}dy \qquad (2.10)$$

$$\int_0^\delta \left[u\frac{\partial u}{\partial y}\left(\int_0^y \frac{\partial u}{\partial x}dy\right)\right]dy=\frac{1}{2}\int_0^\delta \left[\frac{\partial\left(u^2\right)}{\partial y}\left(\int_0^y \frac{\partial u}{\partial x}dy\right)\right]dy$$

$$=\frac{1}{2}U^2\int_0^\delta \frac{\partial u}{\partial x}dy-\frac{1}{6}\int_0^\delta \frac{\partial\left(u^3\right)}{\partial x}dy \qquad (2.11)$$

(using integration by parts).

$$\int_0^\delta uU\frac{dU}{dx}dy = \frac{1}{2}\int_0^\delta u\frac{d}{dx}\left(U^2\right)dy = \frac{1}{2}\int_0^\delta \frac{\partial\left(uU^2\right)}{\partial x}dy - \frac{1}{2}U^2\int_0^\delta \frac{\partial u}{\partial x}dy \qquad (2.12)$$

The rhs of (2.9) is also simplified by integration by parts as follows:

$$v\int_0^\delta u\frac{\partial^2 u}{\partial y^2}dy = v\left\{\left[u\frac{\partial u}{\partial y}\right]_0^\delta - \int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy\right\} = -v\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (2.13)$$

(since the term $\left[u\frac{\partial u}{\partial y}\right]_0^\delta$ vanishes for *u=0* at *y=0* and for $\frac{\partial u}{\partial y} = 0$ at $y=\delta$)

Substituting (2.10), (2.11), (2.12) and (2.13) into (2.9) and simplifying, we obtain

$$\frac{1}{2}\int_0^\delta \frac{\partial}{\partial x}\left(u^3\right)dy - \frac{1}{2}\int_0^\delta \frac{\partial}{\partial x}\left(uU^2\right)dy = -v\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (2.14)$$

Writing the lhs of (2.14) compactly, eqn (2.14) becomes

$$\frac{1}{2}\frac{\partial}{\partial x}\int_0^\delta \left(uU^2 - u^3\right)dy = v\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (2.15)$$

or

$$\frac{\rho}{2}\frac{\partial}{\partial x}\int_0^\delta u\left(U^2 - u^2\right)dy = \mu\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (2.16)$$

(using $v = \frac{\mu}{\rho}$)

It should be noted that for values of $y \geq \delta$ we have $\frac{\partial u}{\partial y} = 0$ and for *y=0*, we also have *u=0*. Thus, we can without loss of generality replace $y=\delta$ by $y\to\infty$ as the upper limit of integration, so that (2.16) becomes

$$\frac{\rho}{2}\frac{\partial}{\partial x}\int_0^\delta u\left(U^2 - u^2\right)dy = \mu\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (2.17)$$

By definition, energy thickness, $\delta_3(x)$, is given by (Wieghardt [1])

$$U^3\delta_3(x) = \int_0^\infty u\left(U^2 - u^2\right)dy \qquad (2.18)$$

so that (since the lhs of (2.18) is a function of *x* only), (2.17) can be expressed in the form

$$\frac{\rho}{2}\frac{d}{dx}\left(U^3\delta_3(x)\right) = \mu\int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (2.19)$$

In eqn. (2.19) the lhs represents loss of mechanical energy, while the rhs represents the energy dissipated in friction and converted into heat.

3.0 Application of Parabolic Velocity Profile in the Energy Thickness Equation and Momentum Integral Equation

From the parabolic velocity profile

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad (3.1)$$

we find

$$\frac{\partial u}{\partial y} = U\left(\frac{2}{\delta} - \frac{2y}{\delta^2}\right),$$

so that

$$\left(\frac{\partial u}{\partial y}\right)^2 = U^2\left(\frac{2}{\delta}-\frac{2y}{\delta^2}\right)^2 = U^2\left(\frac{4}{\delta^2}-\frac{8y}{\delta^3}+\frac{4y^2}{\delta^4}\right) \tag{3.2}$$

Substituting (3.2) in the rhs of (2.19) we find

$$\frac{\rho}{2}\frac{d}{dx}\left(u^3\delta_3\right) = \mu\int_0^\infty\left(\frac{\partial u}{\partial y}\right)^2 dy \approx \mu\int_0^\infty U^2\left(\frac{4}{\delta^2}-\frac{8y}{\delta^3}+\frac{4y^2}{\delta^4}\right)dy \tag{3.3}$$

Integrating the *rhs* of (3.3), we find after simplification

$$\frac{\rho}{2}\frac{d}{dx}\left(U^3\delta_3(x)\right) = \mu U^2\left(\frac{4\delta}{\delta^2}-\frac{4\delta^2}{\delta^3}+\frac{4\delta^3}{\delta^4}\right)$$

i.e

$$\frac{\rho}{2}\frac{d}{dx}\left(U^3\delta_3(x)\right) = \frac{4\mu\ u^2}{3\delta} \tag{3.4}$$

or

$$\frac{d(\delta_3)}{dx} = \frac{2}{\rho u^3}x\frac{4\mu u^2}{3\delta} = \frac{8\mu}{3\rho u\delta} \tag{3.5}$$

Now, Karman-Pohlhawsen momentum integral equation for energy thickness, δ_3, is given by

$$\delta_3(x) = \int_0^\delta \frac{u}{U}\left[1-\left(\frac{u}{U}\right)^2\right]dy \tag{3.6}$$

Substituting (3.1) in (3.6) we have

$$\delta_3(x) = \int_0^\delta \left[\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)-\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)^3\right]dy \tag{3.7}$$

$$\delta_3(x) = \int_0^\delta \left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}-\frac{8y^3}{\delta^3}+\frac{8y^4}{\delta^4}-\frac{2y^5}{\delta^5}+\frac{4y^4}{\delta^4}-\frac{4y^5}{\delta^5}+\frac{y^6}{\delta^6}\right)dy \tag{3.8}$$

which on integration and simplification gives

$$\delta_3(x) = \frac{22\delta}{105} \tag{3.9}$$

or

$$\frac{d}{dx}\delta_3(x) = \frac{22}{105}\frac{d}{dx}\delta \tag{3.10}$$

Comparing (3.5) and (3.10), we find

$$\frac{22}{105}\frac{d}{dx}\delta = \frac{8\mu}{3\rho\ u\delta} \tag{3.11}$$

i.e

$$\frac{11}{105}\frac{d(\delta^2)}{dx} = \frac{8\mu}{3\rho\ u} \tag{3.12}$$

or

$$\frac{d(\delta^2)}{dx} = \frac{280}{11}.\frac{\mu}{\rho\ u} \tag{3.13}$$

Integrating (3.13) we get

$$\delta^2 = 25.45\frac{\mu x}{\rho\ u} + C, \tag{3.14}$$

where C is the constant of integration

We note that when $x = 0$, $\delta = 0$ so that $C = 0$. Thus equation (3.14) becomes

$$\delta = 5.044x\sqrt{\frac{\mu}{\rho u x}} \tag{3.15}$$

or

$$\frac{\delta}{x} = \frac{5.044}{(\mathrm{Re}_x)^{1/2}} \tag{3.16}$$

Another way of writing this is

$$\mathrm{Re}_\delta = 5.044(\mathrm{Re}_x)^{1/2} \tag{3.17}$$

4.0 Discussion and Conclusion

In this work, we see that incorporation of a parabolic velocity profile in Wieghardt's definition of energy thickness and in Karman-Pohlhausen momentum integral equation can provide an alternative method of determining the approximate value of the boundary layer thickness for this profile. Comparison of the approximate value of the boundary layer thickness (3.16) or (3.17) with the exact value, $\mathrm{Re}_\delta = 5.0(\mathrm{Re}_x)^{\frac{1}{2}}$, shows that the percentage error for this parameter is about 0.88%. This result is adequate for the present purpose since it closely tends to the exact value [4]. In (3.16), we notice that as the boundary layer flow becomes more laminar (i.e $\mathrm{Re} << 5\times10^5$), the boundary layer thickness increases with the corresponding increase in the percentage error (or the boundary layer thickness is small when the Reynolds number is large and hence decrease in the percentage error). On the other hand, if the boundary layer thickness is written as in (3.17), we observe that Re_δ is large when Re_x is large and vice versa. Finally, it should be noted that this method is applicable to the velocity profile applied to laminar boundary layer flow.

References

[1] Wieghardt, K. (1948). On an energy equation for the calculation of laminar boundary layers. Ing. Arch., 16, p. 231.

[2] Schlichting, H. (1968). Boundary layer theory (6th ed.). McGraw-Hill, New York.

[3] Pohlhausen, K. (1921). Zur Naherungsweisen Integration der Differentialgleichung der laminrer Grenzshicht. ZAMM, 1, 252 – 368.

[4] Blasius, H. (1908). Grenzchichten in Flussigluiten mit Klener reibung. Zeitschrift fur angewande Mathematik and Physik, 56, English Translation NACATM, No. 1256.

[5] Prandtl, L. (1925). Bericht uber Untersuchungen Zur ausgebildeten Turbulenz. Z. Angew. Math. Mech., 5(2), p. 136.

[6] Craft, K. L. and Lowell, R. P. (2009). A boundary layer model for submarine hydrothermal heat flows at on-axis and near-axis Regions. Geochem. Geophys. Geosyst. 10(12), Q12012.

[7] Habib, A., Schalau, B., Acikalin, A. and Steinbach, J. (2009). Transient calculation of the boundary layer flow over spills. Chemical Engineering and Technology, 32(2), 306 – 311.

[8] Dorfman, A. (2011). Universal functions in boundary layer theory. Fundamental Journal of Thermal Science and Engineering, 1(1), 1 – 72.

[9] Olsson, R. C. and Turkdogan, E. T. (1966). Radial spread of a liquid stream on a horizontal plate. Nature, 211, 813 – 816.

[10] Mohmoudian, A., and Scales, W. A. (2012). Irregularity excitation associated with charged dust cloud boundary layers. Journal of Geophysical Research, 117, A 02304, p. 11.

[11] Kim, B., and Changhoon, L. (2009). Large-eddy simulation of urban boundary layer flow over complex topologies. American Physical Society, 62nd Annual meeting of the APS Division of Fluid Dhynamics.

[12] Eyo, A. E., Nkem, O. and Ekpenyong, M. E. (2012). Comparison of the exact and approximate values of certain parameters in laminar boundary layer flow using some velocity profiles. Journal of Mathematics Research, 4(5), 1 – 13.

[13] Huguera, F. J. (1994). The hydraulic jump in a viscous laminar flow. J. Fluid Mech., 274, 69 – 92.

[14] Bohr, T., Dimond, P. and Putkaradize, V. (1993). Shallow water approach to circular hydraulic jump. J. Fluid Mech., 254, 635 – 645.