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Stresses and Deformation in a Neo-Hookean Half-Space Deforming Under Anti-Plane Shear Loading

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Abstract

Anti-plane shear deformation of a Neo-Hookean half-space is studied. The analysis of a cylindrical section of the material leads to a single linear partial differential equation for the determination of stresses and displacement. An asymptotic solution of the boundary value problem is sought for in a Sobolev space of order 2. The method gives exact solutions for the case of a halfspace containing a central circular cavity and the case in which the central cavity is patched with a solid inclusion.

1.0 Introduction

The fundamental problem in the analysis of solids deforming under anti-plane shear is the difficulty in arriving at exact and closed form solutions to the boundary value problems resulting there from. The more complicated the compliance function of the material, the more tasking is the analysis.

In literature however, various authors have made a number of contributions towards providing a method of analysis that could readily give closed form solutions for the determination of stresses and displacements at various points of various solids deforming under anti-plane shear loading.

Knowel [1] in his paper on the finite anti-plane shear field near the tip of a crack for a class of incompressible elastic solid, used a 'construction' method to describe the stress distribution in a typical incompressible solid containing an initial crack of a fixed length. Other authors [2 – 5] found knowels method very helpful and applied it in the analysis of shear deformation of various incompressible solids. In each case useful results were obtained.

In [6] an asymptotic approach was used to find the solution of the boundary value problem for the determination of stresses and displacements in an Ogden solid deforming under anti-plane shear to a very good approximation.

In this work, the method of [6] is used to provide an exact closed form solution for the boundary value problem for the anti-plane shear deformation of a Neo-Hookean half space containing central, circular flaws. Two cases are considered. The first is a case in which the flaw is a circular cavity at the centre of the half space at undeformed configuration, while the second case is the situation in which the void is patched prior to deformation. The patch or filling is done with a material such that a perfect bond is created at interface. In otherwords there is no differential motion at interface. In either case the analysis leads to exact closed form solution for the stresses and displacements.

2.0 The Field Equations

Let the deformation that takes the point (X_1, X_2, X_3) of the undeformed configuration to the point (x_1, x_2, x_3) of the deformed configuration of an isotropic, homogeneous, incompressible, elastic solid of Neo-Hookean half space under anti-plane shear be

$$\begin{aligned} x_1 &= X_1 \\ x_2 &= X_2 \\ x_3 &= X_3 + g(X_1, X_3), \end{aligned} \qquad (2.1)$$

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where g is a twice differentiable function of X_1 and X_2 at every point P and its image P^*. The deformation gradient tensor $\overline{F}$ is given in components as

$$\overline{F} = x_{i,R} \tag{2.2}$$

where i = 1, 2, 3 and R = 1, 2, 3

The Caucly-Green Left deformation strain tensor is given by

$$\overline{B} = \overline{F}\,\overline{F}^T \tag{2.3}$$

where

$\overline{F}^T$ is the transpose of $\overline{F}$

Hence the invariants I_1, I_2 and I_3 [7] are given by

$$I_1 = 3 + \left(\frac{\partial g}{\partial x_1}\right)^2 + \left(\frac{\partial g}{\partial x_2}\right)^2 = I_2 \tag{2.4}$$

$$I_3 = 1$$

For incompressible solids the stress tensor [1] is given by

$$\overline{\sigma} = -\rho I + 2W_1\overline{B} - 2W_2\overline{B}^{-1} \tag{2.5}$$

where $W_i = \dfrac{\partial W}{\partial I_i}$, I is the unit tensor ρ is the hydrostatic pressure and W is the strain energy density function of the material.

The equilibrium equation, in the absence of body force [7] is given by

$$\nabla \cdot \overline{\sigma} = 0 \tag{2.6}$$

The Neo-Hookean solid is characterized by the strain energy density function

$$W = \frac{\mu}{2}(I_1 - 3) \tag{2.7}$$

where μ is the shear modulus

Consequently the expressions for the non trivial stress components in cylindrical polar coordinates [6] are given by

$$\tau_{rz} = \mu g_r$$
$$\tau_{\theta z} = \frac{\mu}{r} g_\theta \tag{2.8}$$
$$\tau_{zz} = \mu\left(1 + g_r^2 + \frac{1}{r^2} g_\theta^2\right)$$

where $g_k = \frac{\partial g}{\partial k}$

Substituting (2.8) into (2.6) we have that the nontrivial equation of equilibrium in the absence of body force is given by

$$g_{rr} + \frac{1}{r} g_r + \frac{1}{r^2} g_{\theta\theta} = 0 \tag{2.9}$$

3.0 Boundary Value Problem

3.1 Solid with Central Cavity

Consider a solid in a state of plane strain characterized by the strain energy function (2.7), containing a central circular cavity of radius 'a' and under an anti-plane shear loading. The surface, P is traction free. The boundary conditions are that at infinity the stress approximates that of simple shear. The boundary of the hole r = a is traction free. Hence we have the boundary conditions as

$$g = k\, r \sin\theta \qquad r \to \theta$$
$$g_r = 0 \qquad r = a \tag{3.1}$$

Therefore we need to solve equation (2.9) subject to the boundary conditions (3.1). We seek for an asymptotic solution in $W^{1,2}$ (Sobolev space of order 2) which approximates the solution of the boundary value problem

Now let

$$g = k\left(r + \frac{b}{r} + \frac{c}{r^2}\right)\sin\theta \tag{3.2}$$

where $a \le r < \infty, \qquad 0 \le \theta \le 2\pi$

and b and c are to be determined. k is the magnitude of shear as $r \to \infty$. By this choice of g we see that the first condition of (3.1) is trivially satisfied. Now the choice of b and c must be such that the second condition of (3.1) is satisfied

Hence using (3.1b) in (3.2) we have that

$$b = \frac{a^3 - 2c}{a} \tag{3.3}$$

By substituting (3.3) in (3.2) we obtain

$$g_0 = k\left[r+\frac{a^2}{r}+c\left(\frac{1}{r^2}-\frac{2}{ar}\right)\right]\sin\theta \qquad (3.4)$$

where a is the radius of the central cavity.
Substituting (3.4) into (2.9) we have the error term $\in$(r, θ, c) as

$$\in(r,\theta,c) = \frac{3kc}{r^4}\sin\theta \qquad (3.5)$$

We need to minimize the error $\in$ in $W^{1,2}$. In this space $\in$ is minimum when

$$\frac{d}{dc}\|\in(r,\theta,c)\| = \frac{d}{dc}\sqrt{\int_0^{2\pi}\int_a^{\infty}\left[\in^2+\in r^2+\frac{1}{r^2}\in_\theta^2\right]r dr\, d\theta} = 0 \qquad (3.6)$$

Substituting (3.5) into (3.6) we find that $\in$ is minimum when c = 0. Hence

$$b = a^2 \qquad (3.7)$$

Consequently the function g is given by

$$g = k\left(r+\frac{a^2}{r}\right)\sin\theta, \qquad (3.8)$$

and the stresses are

$$\tau_{rz} = uk\left(1-\frac{a^2}{r^2}\right)\sin\theta \qquad (3.9)$$

$$\tau_{\theta z} = \frac{uk}{r}\left(1-\frac{a^2}{r}\right)\cos\theta \qquad (3.10)$$

$$\tau_{zz} = u\left(1+k^2+\frac{k^2a^4}{r^4}+\frac{2k^2a^2}{r^2}\cos 2\theta\right) \qquad (3.11)$$

3.2 Solid with a Solid Circular Inclusion

Let the circular cavity be filled with a solid inclusion such that there is no differential displacement at the boundary r = a. In this case the boundary conditions become

$$g = k r \sin\theta \qquad r \to \infty$$
$$g = 0 \qquad r = a \qquad (3.12)$$

We use the same approximation as in section 3.1. Here substitution and using the boundary conditions give

$$b = -\left(\frac{a^3+c}{a}\right) \qquad (3.13)$$

Using the same analysis as in the previous section we obtain

$$b = -a^2, \qquad (3.14)$$

So that

$$g = k\left(r-\frac{a^2}{r}\right)\sin\theta \qquad (3.15)$$

and

$$\tau_{rz} = \mu k\left(1+\frac{a^2}{r^2}\right)\sin\theta \qquad (3.16)$$

$$\tau_{\theta z} = \frac{\mu k}{r}\left(r-\frac{a^2}{r}\right)\cos\theta \qquad (3.17)$$

$$\tau_{zz} = \mu\left[1+k^2+\frac{k^2a^4}{r^4}-\frac{2k^2a^2}{r^2}\cos 2\theta\right] \qquad (3.18)$$

4.0 Summary and Conclusion

It is easy to see that the solutions obtained in the analysis here are exact [8]. The advantage of minimizing the error in the Sobolev space norm is that the process minimizes the function as well as its gradient. From the solutions we observe that the maximum stress for the solid with a central cavity occurs when $\theta = 0$ while it occurs when $\theta = \pi/2$ for the case in which the cavity is patched with solid inclusion. This paper also serves to validate the use of this method in analysis [9 – 10].

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