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Response Analysis of Euler-Bernoulli Beam Subjected To Partially Distributed Load

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Abstract

This paper investigates the dynamic behavior of beams most especially the Euler-Bernoulli beam with structural damping coefficient subjected to Partially Distributed Moving Loads. The governing Partial differential equation is solved using analytical-numerical method. It was observed that as damping increases, the resultant solution from transformed equation also increases keeping the fixed length of the beam constant.

1.0 Introduction

The primary aim of this paper is to introduce relatively simple manner the subject of vibration as it applies to vibration of beams traversed by uniformly distributed moving loads. The vibration of this moving load will be considered under trucks and railways bridges. Then, the flexural motion of elastic structure will be analyzed.

Vibration of beams due to a moving loads is a field of interest in mechanical, industrial and civil engineering, vibrations of the kind occur in running, railways, bridges, beam subjected to pressure waves and piping systems subjected to two phases flow. The moving loads may be roughly divided into three groups; moving oscillators, moving mass and moving forces [1- 4]

The vibrations of beams due to a moving arbitrary force was studied in [5] where the effects of beam damping, boundary conditions and the speed of the moving load [5] were considered.

The liners and non-linear vibrations analysis of structural elements such as rods, beams, plates and shells, under the act of travelling masses/forces is of considerable practical importance in Civil and mechanical engineering structured under actual operating conditions.

1.1 Nature of Vibration

Vibration is mostly defined as oscillating motion. Beam is a piece of horizontal structure that is usually supported at both ends. It can be in form of wood, metal or plastic, this is concerned with the theory describing the respond of elastic structure under the influence of partially distributed moving loads. The most obvious example of structure subjected to partially distribute moving loads is railway bridges. Furthermore, there is a form of interaction between the motion of the bridge and suspension of the vehicle. Some load applied statically especially if the riding surface is uneven [6 - 8].

In general, there are two types of theory of flexural motion of elastic structure.

(i) the thick-structure theory which account for the effect of shear deformation and rotatory inertia while,

(ii) the classical thin structures neglects the effects of shear information and rotatory inertial. This paper has therefore been motivated by the above stated observation. An investigation into the dynamic response of a Bernoulli-beam resting on a Winkler foundation subjected to partially distributed moving load is presented, the resulting coupled partial differential equation is solved using finite difference method [9 - 13].

1.2 Damping and Undamping Vibration

Damping is the process by which vibration steadily diminishes in amplitude while the beam is non-prismatic.

Undamped vibration is the dynamic response of a simply supported Rayleigh beam caring partially distributed moving loads. In this case only the rotatory inertial is taken into consideration. At the end, it was discovered that, as the value of **r** (radius of gyration) increases, the amplitude of the deflection also increases.

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1.3 Moving Loads

Loads are generally forces acting on a structure. When loads act on a structure, they produce stress and deformations. Loads could be of constant or variable magnitudes. They could also be static or dynamic. Dynamic loads are generally functions of time which may or not continually change position. Dynamic loads which continually change their position are called moving loads. Examples include trucks, trains and cars.

Also, it is known that stresses under the moving loads are much greater than those under the same loads applied statically, especially if the riding surface is uneven.

Below are types of vibration of forces:

(i) Human induced forces: This comes under the heading of moving loads and important sources of dynamic excitation.

(ii) Pedestrians walking or running across footbridges have been known to induce enough vibration to be alarming

(iii) Vibration of light floors caused by heel impact in normal usage can be disturbing to occupant of buildings.

1.4 Moving Load Problems

The problem of carrying out a dynamic of structures under moving loads is known as moving load problem, such moving load problem are of practical importance. The most obvious examples of structure subjected to moving load is highway and railways bridges.

There are two classes of moving loads problem, the first class consist of problem involving concentrated forces (masses) moving with a specified velocity, while the other class deals with the problem of Vibration analysis of structure due to Partially distributed moving forces (masses) [5,14 - 18].

1.5 Governing Equation

Consider a non-prismatic Euler-Bernoulli Beam of length L resting on Winkler foundation and traversed by uniform partially distributed moving mass. The resulting vibrational behavior of this system is described by the following equation.

$$\frac{EI}{M_1} w_{xxxx}(x,t) + \frac{\lambda_0}{m_i} wt^{(x,t)} + \frac{k}{m_i} w_u(x,t) + w_{tt}(x,t) = \frac{1}{\in}\left[-m_g - \frac{md^2w}{dt^2}\left\{H\left(x - \in + \frac{\in}{2}\right) - H\left(x - \in - \frac{E}{2}\right)\right\}\right]$$

where $\gamma_0(x,t) = \frac{1}{\in}\left[-m_g - m\frac{d^2w}{dt^2}\left\{H\left(\in + \frac{\in}{2}\right) - H\left(x - \in + \frac{\in}{2}\right)\right\}\right]$

where $w(x,t) = \sum_{i=1}^{\infty} X_i(x), \lambda_j(t)$

and

$$\gamma_0(x,t) = \sum_{i=1}^{\infty} X_i(x), \lambda_j(t)$$

Over the condition, $w(0,t) = w(\pi,t) = w_{xx}(0,t) = w_{xx}(\pi,t) = 0.$

$$\frac{EI}{M_1} W_{xxxx}{}^{(x,t)} + \frac{\lambda_0}{m_i} W_t{}^{(x,t)} + \frac{K}{M_i} w\ (x,t) + W_{tt}(x,t) = \frac{1}{\in}\left[-M_g - m\frac{d^2w}{dt^2}\left\{H\left(x - \in + \frac{\in}{2}\right) - H\left(x - \in - \frac{E}{2}\right)\right\}\right] \quad (3.0)$$

Where $W(x,t) = \sum_{i=1}^{\infty} X_i(x), \gamma_j(t)$

λ_0 Is viscous Damping Co-efficient

K is Coefficient of Winkler foundation (force per length square).

M_i is mass of the beam.

X is the axial coordinate

t is Time

u is velocity

E is Modulus of Elasticity

M is of the load
EI is the flexural rigidity of the beam
I is Moment of inertia
g is Acceleration due to gravity
$W(x,t)$ is the lateral deflection of the beam measured upward from its equilibrium position when unloaded.
The boundary and initial conditions for the problem are;

$$W(x,t)=W_x(x,t)=0\, at\ \ x=0\, or\, x=e$$
$$W(x,t)=W_{xx}(x,t)=0\, at\ \ x=0\, or\, x=e$$
$$W_{xx}(x,t)=W_{xxx}(x,t)=0\, at\ \ x=0\, or\, x=e$$
$$W_x(x,t)=W_{xxx}(x,t)=0\, at\ \ x=0\, or\, x=e$$

Furthermore, the total derivative $W_{tt}(x,t)$ which appears in equation (3.0) is defined as

$$W_{tt}(x,t)=W_{tt}(x,t)+2VW_{st}(x,t)+v^2W_{ss}(x,t)$$

Where V is the constant velocity of the mass which is defined such that $\in = vt+\frac{\in}{2}$

$$\frac{EI}{M_1}W_{xxxx}{}^{(x,t)}+\frac{\lambda_0}{M_i}W_t{}^{(x,t)}+\frac{K}{M_i}w\ (x,t)+W_{tt}(x,t)=\frac{1}{\in}\left[-M_g-m\frac{d^2w}{dt^2}\left\{H\left(x-\in+\frac{\in}{2}\right)-H\left(x-\in-\frac{E}{2}\right)\right\}\right] \quad (3.1a)$$

Establishing the differentials in the governing equation

$$\left.\begin{aligned} w_t(x,t)&=\sum_{i-1}^{\infty}X_i(x)\overset{0}{\gamma}_j(t)\\ w_{tt}(x,t)&=\sum_{i-1}^{\infty}X_i(x)\overset{0}{\gamma}_j(t)\\ w_{xx}(x,t)&=\sum_{i-1}^{\infty}X_i^{11}(x)\gamma_j(t)\\ w_{xt}(x,t)&=\sum_{i-1}^{\infty}X_i^{i}(x)\overset{0}{\gamma}_j(t)\\ w_{txx}(x,t)&=\sum_{i-1}^{\infty}X_i^{ii}(x)\overset{0}{\gamma}_j(t)\\ w_{xxx}(x,t)&=\sum_{i-1}^{\infty}X_i^{iv}(x)\gamma_j(t)\end{aligned}\right\} \quad (3.1b)$$

Substituting (3.1b) into equation (3.1a)

$$\frac{EI}{Mi}\sum_{i=1}^{\infty}X_i^{iv}(x)\gamma_j(t)+\frac{\lambda_0}{Mi}\sum_{i=1}^{\infty}X_i(x)\overset{0}{\gamma}_j(t)+\frac{K}{Mi}\sum_{i=1}^{\infty}X_i(t)\gamma_j(t)+\sum_{i=1}^{\infty}X_i^{v}(x)\overset{\infty}{\gamma}_j(t)=\gamma_0(x,t)=\sum_{i=1}^{\infty}\gamma_{ij}X_i(x)$$

$$=\sum_{i=1}^{\infty}\left[\frac{EI}{Mi}X_i^{iv}(x)\gamma_j(t)+\frac{\lambda_0}{Mi}\sum_{i=1}^{\infty}X_i(x)\overset{0}{\gamma}(t)+\frac{K}{M_i}X_i(x)\gamma_j(t)-\gamma_{ij}X_i(x)\right] \quad (3.2)$$

$$\frac{EI}{M_1}X_i^{iv}(x)\gamma_j(t)+\frac{\lambda_0}{M_i}X_i(x)\overset{0}{\gamma}_j(t)+\frac{K}{Mi}X_i(x)\gamma_j(t)+X_i(x)\gamma_j(t)=X_{iki}X_i(x) \quad (3.3)$$

Considering the (R.H.S) of the equation (3.3)

$$H(x,\in)=H\left[x-\in+\frac{E}{2}\right]-H\left[x-\in+\frac{E}{2}\right]$$

Putting

H(x) is the Heaviside unit function usually defined as $H(x)=\begin{cases}0, x<0\\ 1, x>0\end{cases}$

Note that: $\frac{d^2w}{dt^2}=W_{tt}(x,t)+2VW_{XE}(x,t)+V^2W_{xx}$

$$i=\frac{1}{\in}\left[-Mg-m\left(W_{tt}(x,t)+2VW_{XE}(x,t)+W_{xx}\right)\right]H(x,\in)=\sum_{ii=1}^{\infty}\gamma_{ij}(t)X_i(x)$$

$$i=\frac{1}{\in}\left[-Mg-m\left(\sum_{i=1}^{\infty}X_i(x)\gamma_j(t)+2V\sum_{i=1}^{\infty}X_i^i(x)\gamma_j(t)+V^2\sum_{i=1}^{\infty}X_i^{ii}(x)\gamma_j(t)\right)\right]H(x,\in)$$

$$=\sum_{i=1}^{\infty}\gamma_{ij}(t)X_i(x)$$

$$i=\frac{1}{\in}\left[\begin{array}{l}-MgH(x,\in)-MH(x,\in)\left[\sum_{i=1}^{\infty}\left(\gamma_j(t)X_i(x)+2V_i\gamma_{ij}(t)X_i^i(x)+V^2\gamma_j(t)X_i^{ii}(x)\right)\right]\\ \qquad\qquad\qquad\qquad\qquad\qquad=\sum_{i=1}^{\infty}\gamma_{ij}[HX_i(x)]\end{array}\right] \qquad (3.4)$$

To normalize equation (3.4), multiply it by $X_j(x)$

From equation (3.4) we have,

$$\frac{X_j(x)}{\in}\left[-MgH(x,\in)-MH(x,\in)\left[\sum_{i=1}^{\infty}\left(\gamma_j(t)X_i(x)+2V_i\gamma_j(t)X_i^i(x)+V^2{}_i\gamma_{ij}(t)X_i^{ii}(x)\right)\right]=\sum_{i=1}^{\infty}\gamma_{ij}[HX_i(x)]\right]$$

(3.5)

Taking the integral of both sides of equation (3.5) with respect to x long the length of the beam.

$$\frac{-Mg}{\in}\int_0^L X_j(x)\left[H\left(x-\in+\frac{H}{2}\right)-H\left(x-\in+\frac{E}{2}\right)\right]dx$$

$$\frac{-Mg}{\in}\sum_{i=1}^{\infty}{}^{0}\gamma_j(t)\int_0^L X_j(x)X_i\left[H\left(x-\in+\frac{H}{2}\right)-H\left(x-\in+\frac{E}{2}\right)\right]dx$$

$$\frac{-2MV}{\in}\sum_{i=1}^{\infty}\gamma_j(t)\int_0^L X_j(x)X_i(x)\left[H\left(x-\in+\frac{H}{2}\right)-H\left(x-\in+\frac{E}{2}\right)\right]dx \qquad (3.6)$$

$$=\sum_{i=1}^{\infty}\gamma_j(t)\int_0^L X_j(x)X_i(x)dx$$

The integral above yield

$$\int_0^L X_j(x)\left[H\left(x-\in+\frac{H}{2}\right)-H\left(x-\in+\frac{E}{2}\right)\right]dx=X_j(\in)+\frac{E^2}{24}X_j^{ii}(\in)+\dots \qquad (3.7)$$

$$\int_0^L X_j(x)X_i(x)\left[H\left(x-\in+\frac{H}{2}\right)-H\left(x-\in+\frac{E}{2}\right)\right]dx=X_j(\in)+\frac{E^2}{24}(X_i(\in)X_j^{ii}(\in))^{ii}+\dots \qquad (3.8)$$

$$\int_0^L X_j(x)X_i^1(x)\left[H\left(x-\in+\frac{H}{2}\right)-H\left(x-\in+\frac{E}{2}\right)\right]dx=X_j(\in)+\frac{E^2}{24}(X_i(\in)X_i^i(\in))^{ii}+\ldots \qquad (3.9)$$

$$\int_0^L X_j(x)X_i^{ii}(x)\left[H\left(x-\in+\frac{H}{2}\right)-H\left(x-\in+\frac{E}{2}\right)\right]dx=X_j(\in)X_i(\in)+\frac{E^2}{24}(X_j(\in)X_i^{ii}(\in))^{ii}\ldots \qquad (3.10)$$

$$\sum_{i=1}^{\infty}\gamma_{ij}(t)\int_0^L X_j(x)X_i(x)dx=\gamma_{ij}(t) \qquad (3.11)$$

Substituting equation (3.7) to (3.11) into equation (3.6)
We have

$$\gamma_{ij}(t)=-Mg\left[X_j(\in)+\frac{E^2}{24}X_i^{ii}(\in)\right]-M\sum_{i-1}^{\infty}\overset{0}{\gamma}_j(t)\left\{X_j(\in)X_i^i(\in)+\frac{E^2}{24}X_j(\in)X_i^{iii}(\in)\right\}+$$
$$2X_j(\in)X_i(\in)+X_j(\in)X_i^{ii}(\in)\ \}-2MV\sum_{i=1}^{\infty}\overset{0}{\gamma}_j(t)\left\{X_j(\in)X_i^i(\in)+\frac{E^2}{24}X_j(\in)X_i^{iii}(\in)\right\}+$$
$$2X_i^{ii}(\in)X_j^i(\in)+X_i^{ii}(\in)\ \}-MV^2\sum_{i=1}^{\infty}\lambda_j(t)\{X_i^{ii}(\in)X_j(\in)+$$
$$\frac{E^2}{24}\left[X_i^{iv}(\in)X_j(\in)+2X_i^{ii}(\in)+X_j(\in)\right]\ \} \qquad (3.12)$$

Putting (3.12) into (3.3)

$$\frac{EI}{Mi}X_i^{iv}(x)\gamma_j(t)+\frac{\lambda_0}{Mi}X_i(x)\overset{0}{\gamma}_j(t)+\frac{K}{Mi}X_i(x)\gamma_j(t)+X_i(x)\overset{0}{\gamma}_j(t)=$$
$$X_i(x)\left[-Mg\left[X_j(\in)+\frac{E^2}{24}X_j^{ii}(\in)\right]\right]-M\sum_{i=1}^{\infty}\overset{\infty}{\gamma}_j(t)\left\{X_j(\in)X_i(\in)+\frac{E^2}{24}X_j^{ii}(\in)X_i(\in)+\right.$$
$$2X_j(\in)X_i(\in)+X_j(\in)X_i^{ii}(\in)\}-2MV\sum_{i=1}^{\infty}\overset{0}{\gamma}_j(t)\{X_j(\in)X_i^i(\in)+\frac{E^2}{24}X_j(\in)_j^{iii}X_i(\in)+$$
$$2X_i^{ii}(\in)X_i^i(\in)X_j^{ii}(\in)+X_i^i(\in)X_j^{ii}(\in)\}-MV^2\sum_{i=1}^{\infty}\overset{0}{\gamma}_j(t)\{X_i^{ii}(\in)X_j(\in)+$$
$$\left.\frac{E}{24}\left[X_j(\in)X_i^{iii}(\in)X_j(\in)+2X_i^{iii}(\in)+X_i^i(\in)+X_i^{ii}(\in)\right]\right\} \qquad (3.13)$$

Now we consider the free vibration of an Euler Bernoulli beam under consideration. Thus we have.

$$X_i^{iv}(x)-\gamma^{(iv)}{}_jX_i(x)=0 \qquad (3.14)$$

Where $\gamma^{(iv)}{}_j=\dfrac{MX_j^2}{EI}$ (3.15)

Thus $X_i^{iv}(x)=\gamma^{(iv)}{}_j\ X_i(x)=\dfrac{MX_j^2X_i(x)}{EI}$ (3.16)

Further substituting (3.16) into (3.13) we have

$$\frac{M}{M_i}\gamma(t)X_j^2X_i(x)+\frac{\lambda_0}{M_i}X_i(x)X_j(t)+\frac{k}{M_i}X_i(x)\overset{\infty}{\gamma}_j(t)=$$

$$X_i(x)\left[-Mg\left(X_j(\in)+\frac{E^2}{24}X_j^{ii}(\in)\right)-M\sum_{i=1}^{\infty}\overset{0}{\gamma}(t)\left\{X_j(\in\right)X_i(\in)+\frac{E^2}{24}X_j^{ii}(\in)X_i(\in)+\right.$$

$$2X_j(\in)X_i(\in)+X_j(\in)X_i^{ii}(\in)\}-2MV\sum_{i=1}^{\infty}\overset{0}{\gamma}_j(t)\left\{X_j(\in)X_i^{i}(\in)+\frac{E^2}{24}X_j^{i}(\in)X_j^{iii}(\in)+\right.$$

$$2X_i^{ii}(\in)X_i^{i}(\in)X_j(\in)+X_i^{ii}(\in)X_j^{ii}(\in)\}-MV^2\sum_{i=1}^{\infty}\overset{0}{\gamma}_j(t)\{X_i^{ii}(\in)X_j(\in)+$$

$$\left.\frac{E}{24}\left[X_j(\in)X_i^{iv}(\in)X_j(\in)+2X_i^{iii}(\in)+X_i^{i}(\in)X_j^{ii}(\in)\right]\right\}$$

Divide through By $X_i(x)$

$$\frac{M}{M_i}\gamma_j(t)X_j^2(x)+\frac{\lambda_0}{M_i}\gamma_j(t)+\overset{\infty}{\gamma}_j(t)=Mg\left[X_j(\in)+\frac{E^2}{24}X_j^{ii}(x)\right]$$

$$-M\sum_{i=1}^{\infty}\overset{\infty}{\gamma}_j\left\{X_j(\in)X_i(\in)+\frac{E^2}{24}\left(X_j^{ii}(\in)X_i(\in)+2X_j(\in)X_j(\in)+X_j(\in)X_i^{ii}(\in)\right)\right\}$$

$$-2MV\sum_{i=1}^{\infty}\overset{\infty}{\gamma}_j(t)X_j\left\{(\in)X_i^{i}(\in)+\frac{E^2}{24}\left(X_j(\in)X_i^{iii}(\in)+2X_i^{ii}(\in)X_i^{i}(\in)X_j^{i}(\in)\right)\right.$$

$$\left.+X_i^{i}(\in)X_j^{ii}(\in)\right\}-MV^2\sum_{i=1}^{\infty}\gamma_j(t)\left\{X_i^{ii}(\in)X_j(\in)+\frac{E}{24}X_i^{iv}(\in)X_j(\in)\right.$$

$$\left.+2X_i^{iii}(\in)+X_i^{i}(\in)+X_i^{ii}(\in)X_j^{ii}(\in)\right\} \qquad (3.17)$$

1.6 The Solution of the Coupled Linear Differential Equation with Prescribed Boundary Conditions

Solving the set of coupled second order differential equation of (3.17) will yield the values of $\gamma_j(t)^s$, we proceed to the next step for solution.

The solve the coupled linear second order differential equation we consider a simply supported beam for which the boundary conditions are given as:

$$w(o,t)=w^{ii}(o,t)=0 \, and \, w(\lambda,t)=w^{ii}(\pi,t)=0 \qquad (3.18)$$

For these condition. It is known that the normalized defection curves $X_j(x)$ is defined by

$$X_j(x)=\sqrt{\frac{2}{L}}Sin\left(\frac{j\pi x}{L}\right) \quad j=1,2.....$$

$$X_j(x)=\sqrt{\frac{2}{L}}Sin\left(\frac{j\pi x}{L}\right) \quad i=1,2..... \qquad (3.19)$$

In order to obtain a set of exact governing differential equation for simply supported beam we substitutive equation (3.19) into (3.6) and obtained

$$\frac{-mg}{\in}\int_0^1\sqrt{\frac{2}{1}}Sin\left(\frac{j\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$\frac{-M}{\in}\sum_{l=i}^{\infty}\gamma_j(t)\int_0^1\sqrt{\frac{2}{1}}Sin\left(\frac{j\pi x}{l}\right)\left(\sqrt{\frac{2}{1}}Sin\left(\frac{i\pi x}{l}\right)\right)^1\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$-\frac{2MV}{\in}\sum_{l=i}^{\infty}\gamma_i(t)\int_0^1\sqrt{\frac{2}{1}}Sin\left(\frac{j\pi x}{l}\right)\left(\sqrt{\frac{2}{1}}Sin\left(\frac{i\pi x}{l}\right)\right)^1\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$-\frac{MV^2}{\in}\sum_{l=i}^{\infty}\gamma_j(t)\int_0^1\sqrt{\frac{2}{1}}Sin\left(\frac{j\pi x}{l}\right)\left(\sqrt{\frac{2}{1}}Sin\left(\frac{i\pi x}{l}\right)\right)^{ll}\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$\sum_{l=i}^{\infty}\gamma_{ij}(t)\int_0^1\sqrt{\frac{2}{1}}Sin\left(\frac{j\pi x}{l}\right)\left(\sqrt{\frac{2}{1}}Sin\left(\frac{i\pi x}{l}\right)\right)dx \qquad (3.20)$$

$$-\frac{mg}{\in}\sqrt{\frac{2}{1}}Sin\left(\frac{j\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$-\frac{2M}{\in}\sum_{l=i}^{\infty}\gamma_j(t)\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Sin\left(\frac{i\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$-\frac{2MV}{L\in}\sum_{l=i}^{\infty}\gamma_j(t)\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Cos\left(\frac{i\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$-\frac{4MV}{L\in}\sum_{l=i}^{\infty}\gamma_i(t)\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Cos\left(\frac{i\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$-\frac{2MV^2i^2\pi^2}{L^3}\sum_{l=i}^{\infty}\gamma_j(t)\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Cos\left(\frac{i\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$\frac{2}{l}\sum_{l=i}^{\infty}\gamma_j^i(t)\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Sin\left(\frac{i\pi x}{l}\right)dx \qquad (3.21)$$

Evaluating the integrals in (3.21)

$$A=\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Sin\left(\frac{i\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$B=\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Sin\left(\frac{i\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$C=\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Cos\left(\frac{i\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$D=\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Sin\left(\frac{i\pi x}{l}\right)\{H(x-\Sigma+\tfrac{\in}{2})-(H(x-\Sigma-\tfrac{\in}{2})\}dx$$

$$E=\int_0^1 Sin\left(\frac{j\pi x}{l}\right)Sin\left(\frac{i\pi x}{l}\right)dx=\begin{cases}1, & i=j\\ 0, & t\neq j\end{cases}$$

We have

$$A=2Sin\left(\frac{j\pi x}{l}\right)Sin\left(\frac{i\pi x}{l}\right) \qquad (3.22a)$$

$$B=\frac{1}{i-j}Cos\frac{\pi\in}{l}(i-j)Sin\frac{\pi\in}{2l}(i-j)-\frac{1}{i+j}\cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i-j) \quad (3.22b)$$

$$C=\frac{1}{i+j}Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i-j)-\frac{1}{i-j}Sin\frac{\pi\in}{l}(i-j)Sin\frac{\pi\in}{2l}(i-j) \quad (3.22c)$$

$$D=\frac{2}{i+j}Sin\frac{\pi\in}{l}(i-j)Cos\frac{\pi\in}{2l}(i-j)-2Sin\frac{\pi}{l}\left(M+\frac{\in}{2}\right)(i+j)Cos\frac{\pi}{l}(i+j) \quad (3.22d)$$

Substituting (3.22a - 3.22d) into equation (3.21)
We obtained

$$\gamma_{ij}=-\frac{mg}{j\pi\in}\sqrt{8l\sin\left(\frac{j\pi x}{l}\right)\sin\left(\frac{j\pi x}{l}\right)}-\frac{2M}{\pi\in}\sum_{i=l}^{\infty}\gamma(t)\left(\frac{l}{i-j}\right)$$

$$Sin\frac{\pi x}{2l}(i-l)Cos\left(\frac{\pi\in}{l}\right)(i-j)+\frac{2m}{\in\pi}\sum\ddot{\gamma}_j(t)Cos\frac{\pi\in}{l}(i+j)Sin\left(\frac{\pi\in}{2l}\right)(i+j)$$

$$-\frac{2MV}{\in}\sum\ddot{\gamma}_j(t)\left(\sqrt{\frac{2}{l}}\right)Sin\frac{\pi\in}{l}(i+j)Sin\left(\frac{\pi\in}{2l}\right)(i-l)+\frac{1}{(i+j)}\left(Sin\frac{\pi\in}{l}(i+j)Sin\left(\frac{\pi\in}{2l}\right)(i-l)\right)$$

$$+\frac{MV^2}{\in}\left(\frac{i\pi}{l}\right)\sum_{i=l}^{\infty}\gamma_j(t)\left(\sqrt{\frac{2}{l}}\left(\frac{1}{i+j}\right)\right)\left(Sin=\frac{\pi\in}{l}(i+j)Cos\left(\frac{\pi\in}{l}\right)(i+j)\right)$$

$$+Cos\left(\frac{\pi\in}{l}\right)(i+j)Sin\left(\frac{\pi\in}{2l}\right)(i+j)i=1,2,3,\ldots l\neq j \quad (3.23)$$

By replacing the R.H.S of (3.17) with the R.H.S of (3.23) we finally obtained

$$M\gamma_j(t)X_j^2+\lambda_0\gamma(t)+M_l\gamma_j(t)=-\frac{Mg}{M_l j\pi\in}\sqrt{8l}Sin\left(\frac{j\pi\in}{l}\right)Sin\left(\frac{i\pi\in}{l}\right)$$

$$-\frac{2M}{\in\pi}\sum_{i=l}^{\infty}\gamma_j(t)\frac{i}{(i-j)}\left(Cos\frac{\pi\in}{l}(i-j)Sin\left(\frac{\pi\in}{2l}\right)(i-j)\right)+\frac{2M}{E\pi}\sum\gamma_j(t)\frac{i}{i+j}Cos\left(\frac{\pi\in}{l}\right)(i+j)$$

$$Sin\frac{\pi\in}{2l}(i+j)-\frac{2MV}{\in}\sum\ddot{\gamma}(t)\left(\sqrt{\frac{2}{l}}\right)Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{l}(i+j)+\frac{1}{(i+j)}$$

$$Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)+\frac{MV^2}{\in}\left(\frac{i\pi}{l}\right)\sum\gamma_j(t)\sqrt{\frac{2}{l}}\left(\frac{i}{(i+j)}\right)$$

$$Sin\frac{\pi\in}{l}(i+j)\ Cos\frac{\pi\in}{2l}(i+j)Cos\left(\frac{\pi\in}{l}(i+j)\right)Sin\left(\frac{\pi\in}{2l}(i+j)\right)i=1,2,3,i\neq j \quad (3.24)$$

1.7 Numerical Solution

$$\gamma_j(t)=\frac{\gamma_{j+1}-\gamma_{j-1}}{2h} \quad (3.25)$$

$$\gamma_j(t)=\frac{\gamma_{j+1}-2\gamma_j+\gamma_{j-1}}{h^2} \quad (3.26)$$

Substituting (3.25), (3.26) into equation (3.24)

$$M\gamma_j(t)X_j^2+\lambda_0\left(\frac{\gamma_{j+1}-\gamma_{j-1}}{2h}\right)+K\gamma(t)+M_1\left(\frac{\gamma_{j+1}-2\gamma_j+\gamma_{j-1}}{h^2}\right)=$$

$$-\frac{Mg}{M_l j\pi\in}\sqrt{8l}Sin\left(\frac{i\pi\in}{l}\right)Sin\left(\frac{j\pi\in}{i}\right)-\frac{2m}{\in\pi}\sum\left(\frac{\gamma_{j+1}-2\gamma_j+\gamma_{j-1}}{h^2}\right)\frac{1}{(i-j)}$$

$$\left(Cos\frac{\pi\in}{l}(i-j)Sin\frac{\pi\in}{2l}(i-j)\right)+\frac{2m}{\in\pi}\sum\left(\frac{\gamma_{j+1}-2\gamma_j+\gamma_{j-1}}{h^2}\right)\frac{1}{(i-j)}Cos\frac{\pi\in}{l}(i+j)$$

$$Sin\frac{\pi\in}{2l}(i+j)-\frac{2MV}{\in}\sum\left(\frac{\gamma_{j+1}-\gamma_{j-1}}{2h}\right)\left(\sqrt{\frac{2}{l}}\right)Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)+\frac{1}{(i+j)}$$

$$\left(Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{x}(i+j)\right)+\frac{MV^2}{\in}\sum\left(\frac{i\pi}{l}\right)\sum_{i=1}^{\infty}\gamma_j\sqrt{\frac{2}{l}}\left(\frac{1}{i+j}\right)Sin\frac{\pi\in}{l}(i+j)$$

$$Cos\frac{\pi\in}{2l}(i+j)+Cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)i=123...i\neq j \qquad (3.27)$$

To solve this and re-arranging, we solve the (L.H.S) and (R.H.S) resp. then multiply (LHS) by h^2.

$$=h^2M\gamma_jX_j^2+\frac{\lambda_0h}{2}(\gamma_{j+l}-\gamma_{j-l})+h^2K\gamma_j+M_i\gamma_{j+l}-2\gamma_j+\gamma_{j-l} \qquad (3.28)$$

$$\left(M_l+\frac{h}{2}\lambda o\right)\gamma_{j+1}+\left(M-\frac{h}{2}\lambda_0\right)\gamma_{j-1}-\left(2M-h^2mX_j^2-h^2k\right)\gamma_j=$$

$$-\frac{Mg}{M_l j\pi\in}\sqrt{8l}Sin\left(\frac{i\pi\in}{l}\right)Sin\left(\frac{j\pi\in}{i}\right)-\frac{2m}{\in\pi}\sum\left(\frac{\gamma_{j+1}-2\gamma_j+\gamma_{j-1}}{h^2}\right)\frac{1}{(i-j)}$$

$$\left(Cos\frac{\pi\in}{l}(i-j)Sin\frac{\pi\in}{2l}(i-j)\right)+\frac{2m}{\in\pi}\sum\left(\frac{\gamma_{j+1}-2\gamma_j+\gamma_{j-1}}{h^2}\right)\frac{1}{(i-j)}Cos\frac{\pi\in}{l}(i+j)$$

$$Sin\frac{\pi\in}{2l}(i+j)-\frac{2MV}{\in}\sum\left(\frac{\gamma_{j+1}-\gamma_{j-1}}{2h}\right)\left(\sqrt{\frac{2}{l}}\right)Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)+\frac{1}{(i+j)} \qquad (3.29)$$

$$\left(Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{x}(i+j)\right)+\frac{MV^2}{\in}\sum\left(\frac{i\pi}{l}\right)\sum_{i=1}^{\infty}\gamma_j\sqrt{\frac{2}{l}}\left(\frac{1}{i+j}\right)Sin\frac{\pi\in}{l}(i+j)$$

$$Cos\frac{\pi\in}{2l}(i+j)+Cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)i=123...i\neq j$$

From RHS of Equation (3.29), Multiply by h^2.

$$-\frac{Mh^2g}{M_i j\pi\in}\sqrt{8l}Sin\left(\frac{j\pi\in}{l}\right)Sin\left(\frac{i\pi\in}{2l}\right)-\frac{2M}{\in\pi}\sum_{i=1}^{\infty}\left(\gamma_{j+1}-2\gamma_j+\gamma_{j-1}\right)\left(\frac{1}{i-j}\right)Cos\frac{\pi\in}{l}(i-j)$$

$$Sin\left(\frac{\pi\in}{2l}\right)(i-j)\}+\frac{2M}{\in\pi}\sum_{i=1}^{\infty}\left(\gamma_{j+1}-2\gamma_j+\gamma_{j-1}\right)\left(\frac{1}{i+j}\right)Cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{l}(i+j)$$

$$\frac{hMV}{\in}\Sigma\left(\gamma_{j+1}-\gamma_{j-1}\right)\left(\sqrt{\frac{2}{l}}\right)\{Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{l}(i+j)+\left(\frac{1}{i-j}\right)Sin\frac{\pi\in}{l}(i+j)$$

$$Sin\frac{\pi\in}{2l}(i+j)\}+\frac{MV^2h^2}{\in}\left(\frac{i\pi}{\in}\right)\sum_{i=1}^{\infty}\gamma_j\sqrt{\frac{2}{l}}Sin\frac{\pi\in}{l}Sin\frac{\pi\in}{l}(i+j)+Sin\left(\frac{\pi\in}{l}(i+j)Sin\left(\frac{\pi\in}{l}(i+j)\right)\right)$$

$$Sin\left(\frac{\pi\in}{l}(i+j)Sin\left(\frac{\pi\in}{l}(i+j)\right)\right) \qquad (3.30)$$

Re-arrange equation (3.30) to get

$$-\frac{Mh^2g}{M_i j\pi\in}\sqrt{8l}Sin\left(\frac{j\pi\in}{l}\right)Sin\left(\frac{i\pi\in}{2l}\right)+\left\{\frac{2M}{\in\pi}\left(\frac{1}{i-j}\right)Cos\left(\frac{i\pi\in}{l}\right)Sin\left(\frac{i\pi\in}{2l}\right)+\frac{2M}{\in\pi}\left(\frac{1}{i+j}\right)\right\}$$

$$Cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)-\frac{hMVi}{\in}\left[\sqrt{\frac{2}{l}}\right]Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)+\left(\frac{1}{i+j}\right)Sin\frac{\pi\in}{l}(i+j)$$

$$in\frac{\pi\in}{2l}(i+j)\}\gamma_{j+i}+\left\{\frac{4M}{\in\pi}\left(\frac{1}{i-j}\right)Cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i-j)-\frac{4M}{\in\pi}\left(\frac{1}{i+j}\right)Cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)\right.$$

$$\frac{MV^2h^2\pi}{\in l}\left(\sqrt{\frac{2}{l}}\right)\left\{\frac{1}{i+j}Sin\frac{\pi\in(i+j)}{l}Cos\frac{\pi\in(i-j)}{2l}(i+j)+Cos\frac{\pi\in(i+j)}{l}+\right.$$

$$Sin\frac{\pi\in}{x}(i+j)\}\gamma_j+\left\{\frac{-2M}{\in\pi}\left(\frac{1}{i-j}\right)Cos\frac{\pi\in}{l}(i-j)Sin\frac{\pi\in}{2l}(i-j)\right\}+\frac{2M}{\in\pi}\left(\frac{1}{i+j}\right)Cos\frac{\pi\in}{l}(i+j)$$

$$Sin\frac{\pi\in}{x}(i+j)+\frac{hMVi}{\in}\left(\sqrt{\frac{2}{l}}\right)Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)+\left(\frac{1}{i+j}\right)Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)\}\gamma_{j-1}$$

$$=\left(M_1+\frac{h}{2}\lambda_0\right)\gamma_{j+1}\left(M_1-\frac{h}{2}\lambda_0\right)\gamma_{j-1}-(2M-h^2mX^2{}_j-h^2k)\gamma_j \qquad (3.31)$$

Equate equation (3.29) to (3.31)

$$\left(M_1+\frac{h}{2}\lambda_0\right)\gamma_{j+1}\left(M_1-\frac{h}{2}\lambda_0\right)\gamma_{j-1}-(2M-h^2mX^2{}_j-h^2k)\gamma_j=\frac{Mh^2g}{M_l j\pi\in}\sqrt{8l}$$

$$Sin\frac{j\pi\in}{l}Sin\frac{i\pi\in}{2l}+\left\{-\frac{2M}{\in\lambda}\left(\frac{1}{i-j}\right)Cos\frac{\pi\in}{l}(i-j)Sin\frac{\pi\in}{2l}(i-j)\right\}+\frac{2M}{\in\pi}\left(\frac{1}{i+j}\right)Cos\frac{\pi\in}{l}(i+j)$$

$$Sin\frac{\pi\in}{2l}(i-j)-\frac{hmvi}{\in}\left(\sqrt{\frac{2}{l}}Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)\right)+\left(\frac{1}{j+j}\right)Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)\}\gamma_{j+1}$$

$$\frac{4M}{\in\pi}\left(\frac{1}{i-j}\right)\left[Cos\frac{\pi\in}{l}(i-j)Sin\frac{\pi\in}{2l}(i-j)-\frac{4M}{\in\pi}\left(\frac{1}{i-j}\right)Cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)\right]+$$

$$\frac{mv^2h^2i\lambda}{\in l}\left(\sqrt{\frac{2}{l}}\right)\left\{\frac{1}{i+j}Sin\frac{\pi\in}{l}(i+j)Cos\frac{\pi\in}{2l}(i+j)+Cos\frac{\pi\in}{l}(i+j)+Sin\frac{\pi\in}{2l}(i+j)\right\}\gamma_j+$$

$$\left\{\frac{-2m}{\pi\in}\left(\frac{1}{i-j}\right)Cos\frac{\pi\in}{l}(i-j)+Sin\frac{\lambda\in}{2l}(i-j)+\frac{-2m}{\pi\in}\left(\frac{1}{i-j}\right)Cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)+\right.$$

$$\frac{hmvi}{\in}\sqrt{\frac{2}{l}}\left(Sin\frac{\pi\in}{l}(i+j)+Sin\frac{\pi\in}{2l}(i+j)+\left(\frac{i}{i+j}\right)\left(Sin\frac{\pi\in}{l}(i+j)+Sin\frac{\pi\in}{2l}(i+j)\right)\right)\gamma_{j-1} \qquad (3.32)$$

Collecting like terms

$$\left\{M_1+\frac{h}{2}\lambda_0+\frac{2m}{\pi\in}\left(\frac{i}{i-j}\right)\left[Cos\frac{\pi\in}{l}(i-j)+Sin\frac{\pi\in}{2l}(i-j)\right]+\frac{2m}{\pi\in}\left(\frac{i}{i+j}\right)Cos\frac{\pi\in}{l}(i+j)+Sin\frac{\pi\in}{2l}(i+j)\right.$$

$$+\frac{hmvi}{\in}\left[\sqrt{\frac{2}{l}}\right]\left(Sin\frac{\pi\in}{l}(i+j)\right)Sin\frac{\pi\in}{2l}(i+j)+\frac{1}{(i+j)}\left(Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)\right)\gamma_{j+1}+$$

$$\left\{M-\frac{h}{l}\lambda_o+\left[\frac{2m}{\pi\in}\left(\frac{i}{i-j}\right)Cos\frac{\pi\in}{l}(i-j)+Sin\frac{\pi\in}{2l}(i+j)\right]-\frac{2m}{\pi\in}\left(\frac{i}{i+j}\right)Cos\frac{\pi\in}{l}(i+j)+Sin\frac{\pi\in}{2l}(i+j)-\right.$$

$$\frac{hmvi}{\in}\sqrt{\frac{2}{l}}Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)+\frac{1}{(i+j)}\left(Sin\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)\right)\gamma_{j+1}+$$

$$\left\{2m+h^2mX_j^2+h^2k+\frac{4m}{\pi\in}\left(\frac{i}{i+j}\right)Cos\frac{\pi\in}{l}(i+j)Sin\frac{\pi\in}{2l}(i+j)-\frac{4m}{\pi\in}\left(\frac{i}{i-j}\right)\left[Cos\frac{\pi\in}{l}(i+j)-Sin\frac{\pi\in}{2l}(i+j)\right]\right.$$

$$\left.-\frac{mv^2h^2i\pi}{\in L}\sqrt{\frac{2}{l}}\left(\frac{i}{i+j}\right)Sin\frac{\pi\in}{l}(i+j)Cos\frac{\pi\in}{l}(i+j)+Cos\frac{\pi\in}{l}(i+j)+Sin\frac{\pi\in}{2l}(i+j)\right\}\gamma_j+$$

$$-\frac{Mh^2g}{M_ij\pi\in}\sqrt{8l}Sin\left(\frac{j\pi\in}{l}\right)Sin\left(\frac{i\pi\in}{2l}\right)\qquad i=1,2,3.\qquad i\neq j \tag{3.33}$$

Conclusion

For the problem concerning the response of an Euler-Bernoulli beam resting on Winkler foundation to partially distributed load. It was observed that the fixed length of the load increase as the amplitude of the deflection increases. Also, the time, t, increases with an increase in the amplitude of the deflection. It was observed that the amplitude of the deflection increases as the value of r increases.

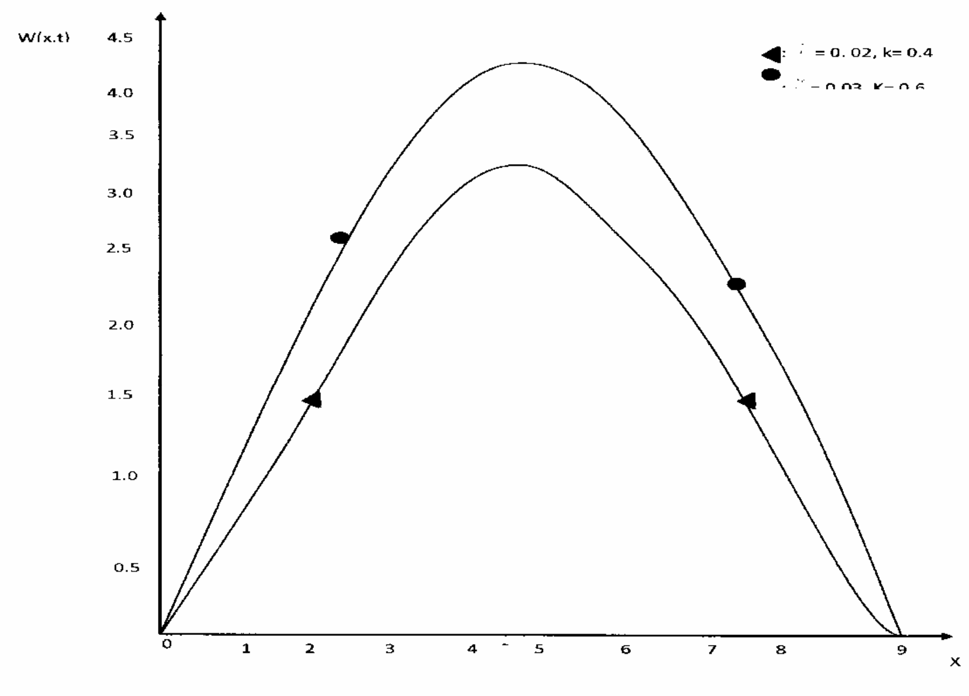

Fig 4.2 Diagram shows deflection of beam for various values of γ and k

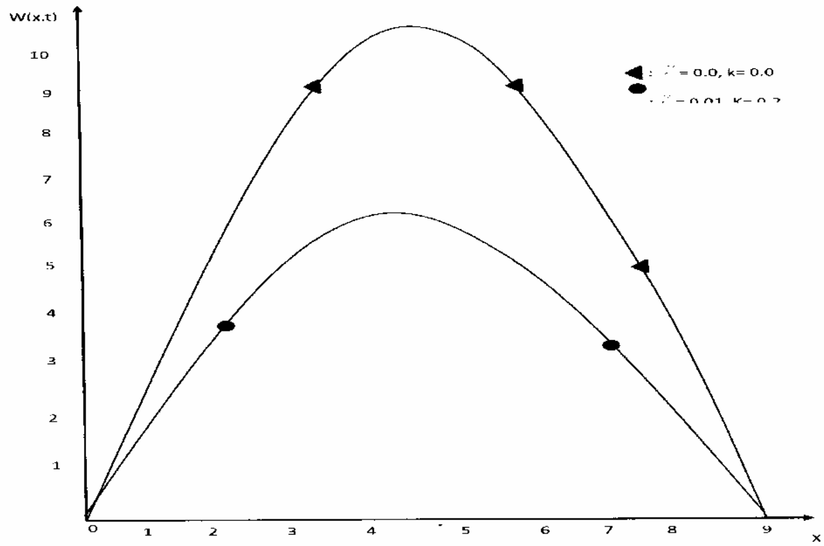

Fig 4.1 Diagram shows deflection of beam for various values of γ and k

References

[1]. Gbadeyan J.A. and Oni, S.T. (1992): Dynamic response to moving concentrated masses of elastic plates on a non-Winkler elastic foundation. Journal of sound and vibration 154(2) Pg 343-358.
[2]. Sadekin and Leiphole H. H. E (1987): On the dynamic of elastic systems with moving concentrated masses. Ingneur Archi 57,223-224
[3]. Stanisic M.M and Hardih J.C. (1969): On the response of beams to an arbitrary number of concentrated moving masses. Journal of the Franklin institute 257,115-123
[4]. Stanisic M, Euler J.A. and Montgomery S.T. (1974): On a theory concerning the dynamical behaviour of structures carryng moving masses. Ingneur archive 43, 295-305
[5]. Esmailzadeh E and Ghorashi M. (1995): Vibration analysis of beams traversed by uniform partially distributed moving masses. Journal of sound and vibration. 184 (4) Pg. 615-828.
[6]. Timoshenko S. young D.H and Weaver W. (1974): Vibration problems in engineering. New York john Wiley, fourth Edition.
[7]. Timoshenko S, (1927): Vibration of bridges. Transactions of the America society of mechanical engineer Vol 74, 53-61
[8]. Inglis C.E. (1934): A Mathematical trease in vibration in Railway bridges Cambridge, Cambridge university Press.
[9]. Mathews J.H. (1987): Numerical methods Englewood cliffs, New Jersey, Prentice-Hall.
[10]. Akin J.E. and Mofid M. (1989): Numerical, solution for response of beams with moving masses. Journal of the structural engineer Vol 115,pg 110-201.
[11]. Timoshenko S. (1986): On the traverse vibration of bar if uniform cross section. Philosophical mag, ser.6, vol.43,pg 125-131.
[12]. Kenny, J. (1984): Steady state Vibration of Beam on Elastic Foundation for a Moving Load. Journal Appl. Mach. vol.76.pg 359-364.
[13]. Usman M.A. (2003): Dynamic response of a Bernoulli beam on winkler foundation under the action of moving partially distributed moving load. NJMA. Vol.16. pg 128-147.
[14] L.FRYBAL (1972): Vibration of solids and structures under moving Loads. Prague. Research institute of Transport.
[15]. Meirovitech I. (1986): Elements of Vibration Analysis, 2nd Edition, McGraw-Hill Book Co, New York.
[16]. Norton M.P (1989): Fundamentals of Noise and Vibration Analysis for Engineers, Cambridge University Press, Cambridge.
[17]. Rao S.S (1990): Mechanical Vibration 2nd Edition, Addison-Wesley Publishing Co. Inc, New York.
[18]. Vierck R.K (1979): Vibration Analysis, International Textbooks Co, Pennsylvania.