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Derivation of Properties of Shifted Legendre Polynomials

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Abstract

These four properties of the Legendre polynomials: Rodrigues' formula, mutual orthogonality, generating function and recurrence formula are discussed in passing. The linear change of variable is employed to transform the Legendre polynomials to the shifted Legendre polynomials. The above properties are then derived in detail for shifted Legendre polynomials. Furthermore, certain theorems are used to prove that the derived Rodrigues' formula and the recurrence relation properties actually satisfy the shifted Legendre equation.

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1.0 Introduction

The Solution of the Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1}$$

with singular points at $x = -1,1$, where n is an integer is of the form

$$y = c_1 P_n(x) + c_2 Q_n(x) \tag{2}$$

Where c_1, c_2 are arbitrary constants, $Q_n(x)$ are Legendre functions and $P_n(x)$ are the Legendre polynomials[1,2] given as

$$P_n(x) = \frac{(2n-1)(2n-3)\dots 1}{n!}\left\{x^n - \frac{n(n-1)}{2(n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + \cdots\right\} \tag{3}$$

The Legendre polynomial $P_n(x)$ can also be expressed by the Rodrigues' formula given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{4}$$

which serves as an aid to establishing further properties of Legendre polynomials such as the first two terms of the $P_n(x)$[1,3].

The Legendre equation(1) can be also put in Sturm-Liouville form

$$P(x)\frac{d^2y}{dx^2} + r(x)\frac{dy}{dx} + q(x)y + \lambda\rho(x)y = 0, \qquad r(x) = \frac{dp(x)}{dx} \tag{5}$$

With $p(x) = 1 - x^2,\ q(x) = 0, \lambda = n(n+1), \rho(x) = 1$ and its natural interval $[-1,1]$. Since $P_n(x)$ are regular at the end point $x = \pm 1$, they must be mutually orthogonal over this interval[1]. That is

$$\int_{-1}^{1} P_n(x)P_k(x)dx = 0 \qquad if\ n \neq k \tag{6}$$

(6) shows that, any two different Legendre polynomials are orthogonal in the interval $-1 < x < 1$ [2].

The generating function for Legendre polynomials which is useful in obtaining their properties[2] is given by

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$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\, t^n \tag{7}$$

This is used with the aid of differentiation to derive the following recurrence formula satisfied by the Legendre polynomials (1)

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x) \tag{8}$$

which is often used for generating Legendre polynomials[1].

However, the Rodrigues' formula, mutual orthogonality, generating function and recurrence formulae properties given by (4), (6), (7) and (8) respectively, have not been derived for the shifted Legendre polynomials. The shifted Legendre polynomials which serve as very good orthogonal basis functions in polynomial approximation especially in the least square sense[4], enjoy wide range of applications in different aspects of mathematical sciences; for instance , in the solution of convolution integral equations[5].

In this work, we derive and discuss these properties for the shifted Legendre polynomials stating some theorems to prove that they actually satisfy the shifted Legendre equation.

2.0 Shifted Legendre Polynomials

Proposition 2.1(Linear Change of Variable)

Consider the inner product [6,7]

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx \tag{9}$$

on the interval $[a, b]$. The map $[-1,1]$ to $[a, b]$ by a simple linear change of variable is of the form

$$t = \alpha + \beta x \tag{10}$$

Particularly,

$$t = \frac{2x-b-a}{b-a} \tag{11}$$

will change $a \le x \le b$ to $-1 \le t \le 1$. The map changes functions $F(t), G(t)$ defined for $-1 \le t \le 1$ into functions

$$f(x) = F\left(\frac{2x-b-a}{b-a}\right) \qquad g(x) = G\left(\frac{2x-b-a}{b-a}\right) \tag{12}$$

defined for $a \le x \le b$.

From the proposition 2.1, the singular points of the Legendre equation are then shifted;

$$(1-t^2)y''(\mathrm{t}) - 2ty'(t) + n(n+1)y(t) = 0 \tag{13}$$

From $x = \pm 1$ to $x = 0,1$.That is

$$t = 2x - 1 \tag{14}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{2} \qquad \Rightarrow \mathrm{y}'(t) = \frac{1}{2}\frac{dy}{dx} \qquad \Rightarrow \mathrm{y}''(t) = \frac{1}{4}\frac{d^2y}{dx^2}$$

Substituting in (13), we get,

$$(x-x^2)y''(x) - (2x-1)y'(x) + n(n+1)y(x) = 0 \tag{15}$$

which is the shifted Legendre equation, whose series solution

$P_n^*(x) = P_n(2x-1)$is called the Shifted Legendre Polynomial[1,2].

3.0 Properties of Shifted Legendre Polynomials

3.1 Rodrigues' Formula

We note from (14) that,

$$\frac{d}{dt} = \frac{1}{2}\frac{d}{dx}$$

...

$$\frac{d^n}{dt^n} = \frac{1}{2^n}\frac{d^n}{dx^n} \tag{16}$$

Putting (14) and (16) in the Rodrigues' formula for Legendre polynomials,

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$$

we get

$$P_n^*(x) = \frac{1}{2^n n!}\frac{1}{2^n}\frac{d^n}{dx^n}[(2x-1)^2-1]^n$$

$$\Rightarrow P_n^*(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^2-x)^n \qquad (17)$$

which is the Rodrigues' formula for shifted Legendre polynomials.

Theorem 3.1

The shifted Legendre polynomial (17) is a solution of the shifted Legendre equation (15)

Proof

Let $u = (x^2-x)^n$

Differentiating with respect to x, we get

$u' = n(2x-x)(x^2-x)^{n-1}$

Multiplying through by (x^2-x) yields

$$(x^2-x)u' - n(2x-1)u = 0 \qquad (18)$$

Differentiating (18)$n+1$ times using the following Leibniz rule for product of functions

$$(fg)^{(n+1)} = \sum_{n=0}^{n+1}\binom{n+1}{k} f^{(n+1-k)}g^{(k)} \qquad (19)$$

Where$\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!}$, the first term of (18) gives

$[(x^2-x)u']^{(n+1)} = (x^2-x)u^{(n+2)} + (2x-1)(n+1)u^{(n+1)} + n(n+1)u^{(n)}$

The second term of (18) gives

$n[(2x-1)u]^{(n+1)} = n\left[(2x-1)u^{(n+1)} + 2(n+1)u^{(n)}\right]$

Combining the two terms, equation (18) becomes

$(x^2-x)u^{(n+2)} + (2x-1)u^{(n+1)} - n(n+1)u^{(n)} = 0$

Multiplying through by -1, we get

$$(x-x^2)u^{(n+2)} - (2x-1)u^{(n+1)} + n(n+1)u^{(n)} = 0 \qquad (20)$$

which recovers the shifted Legendre equation (15) with $u^{(n)}$ now as the independent variable. Since from (17) n is an integer and $u^{(n)}$ has the end points $x = 0,1$, we make the identification

$$u^{(n)}(x) = c_n P_n^* \qquad (21)$$

for some constants c_n. We note that the only term in the expression for the nth derivative of $(x^2-1)^n$ that does not contain a factor (x^2-x) and therefore does not vanish at $x = 1$ is

$$(2x-1)^n n!\,(x^2-x) = u^{(n)}(x) \qquad (22)$$

Putting $x = 1$ in (22), we get

$n! = c_n P_n^*(x)$

Taking $P_n^*(x) = 1$ as the normalization of the equation (17) gives

$c_n = n!$

Thus the Rodrigues' formula for the shifted Legendre polynomial $P_n^*(x)$ is a solution of the shifted Legendre equation (15).

3.2 Mutual Orthogonality

Since $P_n^*(x)$ satisfies the shifted Legendre equation (15), we write

$[(x-x^2)P_n^{*\prime}]' + n(n+1)P_n^* = 0$

Multiplying through and integrating from $x = 0$ to $x = 1$,

$$\int_0^1 P_k^*(x)\left[(x-x^2)P_n^{*'}\right]' dx + \int_0^1 n(n+1)P_k^* P_n^*\, dx = 0$$

Integrating by part, we get

$$-\int_0^1 (x-x^2)\, P_k^{*'} P_n^{*'} dx + \int_0^1 n(n+1)\, P_k^* P_n^* dx = 0 \tag{23}$$

Reversing the roles of n and k yields,

$$-\int_0^1 (x-x^2)\, P_n^{*'} P_k^{*'} dx + \int_0^1 k(k+1)\, P_n^* P_k^* dx = 0 \tag{24}$$

Subtracting (24) from (23), we get

$$\int_0^1 n(n+1)\, P_k^* P_n^* dx - \int_0^1 k(k+1)\, P_n^* P_k^* dx = 0$$

$$[n(n+1) - k(k+1)] \int_0^1 P_n^* P_k^* dx = 0$$

Since $k \neq n \Rightarrow\ [n(n+1) - k(k+1)]\ \neq 0$

That is

$$\int_0^1 P_n^* P_k^* dx = 0 \tag{25}$$

is the orthogonality property of the shifted Legendre polynomial with $w^*(x) = 1$ as the weight function. As a particular case we note that, if we put $k = 0$, we obtain,

$$\int_0^1 P_n^* dx = 0 \qquad for\ n \neq 0$$

Any polynomial of degree k can be written as a linear combination of P_n^* with $n \leq k$ so from (25) polynomial of degree less than n is orthogonal to P_n^*.

3.3 Generating Function

The generating function for shifted Legendre is obtained using proposition 2.1 and the generating function for Legendre polynomials (7). Thus

$$\frac{1}{\sqrt{1-2(2x-1)t+t^2}} = \sum_{n=0}^{\infty} P_n^*(x)\, t^n$$

$$[1-(4x-2)t+t^2]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n^*(x)\, t^n \tag{26}$$

3.4 Recurrence Formula

The first and of course the most important recurrence relation for the shifted Legendre polynomials is obtained using the generating function (26). Differentiating (26) with respect to t, we get

$$\frac{2x-t-1}{\sqrt{[1-(4x-1)t+t^2]^{\frac{3}{2}}}} = \sum_{n=0}^{\infty} nP_n^*\, t^{n-1}$$

Multiplying through by $1-(4x-2)t+t^2$, we get

$$(2x-t-1)\sum_{n=0}^{\infty} P_n^*\, t^n = [1-(4x-2)t+t^2]\sum_{n=0}^{\infty} nP_n^*\, t^{n-1}$$

Expanding and evaluating for powers t^k on both sides gives

$2xP_k^* - P_{k-1}^* - P_k^* = (k+1)P_{k+1}^* - k(4x-2)P_k^* + (k-1)P_{k-1}^*$

$\Rightarrow (k+1)P_{k+1}^* = (2x + 4xn - 2k - 1)P_k^* - kP_{k-1}^*$

Replacing k with n, we get the recurrence formula

$$P_{k+1}^* = \frac{(2x+4nx-2n-1)}{n+1}P_k^* - \frac{n}{n+1}P_{k-1}^* \tag{27}$$

which is used to generate further relation properties of the shifted Legendre polynomials as we shall see below.

Theorem 3.2

The generating function (26) satisfies the shifted Legendre equation (15).

Proof

We first differentiate (26) with respect to x to get

$$\frac{2t}{[1-(4x-2)t+t^2]^{\frac{3}{2}}} = \sum_{n=0}^{\infty} P_n^{*'} t^n \tag{28}$$

Also differentiating (26) with respect to t gives

$$\frac{2x-t-1}{[1-(4x-2)t+t^2]^{\frac{3}{2}}} = \sum_{n=0}^{\infty} nP_n^{*} t^{n-1} \tag{29}$$

Multiplying (28) by $[1-(4x-2)t+t^2]$, we get

$$\sum_{n=0}^{\infty} 2P_n^{*} t^{n+1} = \sum_{n=0}^{\infty} P_n^{*'} t^n - \sum_{n=0}^{\infty} (4x-2)P_n^{*'} t^{n+1} + \sum_{n=0}^{\infty} P_n^{*'} t^{n+2}$$

Equating the coefficient of t^{n+1}, we obtain the recurrence relation

$$2P_n^{*} = P_{n+1}^{*'} - (4x-2)P_n^{*'} + P_{n-1}^{*'} \tag{30}$$

Multiplying (29) by $2t$ and (28) by $(2x-t-1)$ and equating the two results, we obtain

$$(2x-t-1)\sum_{n=0}^{\infty} P_n^{*'} t^n = 2t\sum_{n=0}^{\infty} nP_n^{*} t^{n-1}$$

Expanding and evaluating the coefficient of t^n, we get

$$2xP_n^{*'} - P_{n-1}^{*'} - P_n^{*'} = 2nP_n^{*}$$

which yields the recurrence relation

$$2nP_n^{*} = (2x-1)P_n^{*'} - P_{n-1}^{*'} \tag{31}$$

Adding (30) and (31), we get

$$2nP_n^{*} + 2P_n^{*} = (2x-1)P_n^{*'} - (4x-2)P_n^{*'} + P_{n+1}^{*'}$$

this yields the recurrence relation

$$2(n+1)P_n^{*} = P_{n+1}^{*'} - (2x-1)P_n^{*'} \tag{32}$$

Replacing (32) with$n-1$ for n, results to

$$2nP_{n-1}^{*} = P_n^{*'} - (2x-1)P_{n-1}^{*'} \tag{33}$$

Multiplying (31) by $(2x-1)$, we get

$$2n(2x-1)P_n^{*} = (2x-1)^2P_n^{*'} - (2x-1)P_{n-1}^{*'} \tag{34}$$

Adding (33) and (34) gives,

$$4(x^2-x)P_n^{*'} = 2n(2x-1)P_n^{*} - 2nP_{n-1}^{*}$$

Dividing through by 2 gives the recurrence relation

$$2(x^2-x)P_n^{*'} = n[(2x-1)P_n^{*} - P_{n-1}^{*}] \tag{35}$$

Differentiating both sides with respect to x yields

$$2(x^2-x)P_n^{*''} + 2(2x-1)P_n^{*'} = n\left[\left((2x-1)P_n^{*'} - P_{n-1}^{*'}\right) + 2P_n^{*}\right]$$

Using equation (31), we get

$$2(x^2-x)P_n^{*''} + 2(2x-1)P_n^{*'} - 2n(n+1)P_n^{*}$$

Dividing through by -2 yields the shifted Legendre equation

$$(x-x^2)P_n^{*''} - (2x-1)P_n^{*'} + n(n+1)P_n^{*} = 0$$

with the independent variable as P_n^{*}.

Equations (30)-(32) and (35) can be rearranged to obtain the other recurrence relations of the shifted Legendre polynomial with the last obtained by subtracting equations (32) from (31).

4.0 Conclusion

These properties of the Legendre polynomials: Rodrigues' formula, mutual orthogonality, generating function and the recurrence formulae do exist for the shifted Legendre polynomial and they actually satisfy the shifted Legendre equation.

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