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Some Integration Formulas for Improper Integrals of the First Kind (Involving Exponential Functions) Obtained By Laplace Transform Techniques

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Abstract

In this paper, we have shown how to use the Laplace transform technique to obtain integration formulas for some improper integrals of the first kind involving exponential functions. Some illustrative examples such as;

$$\int_0^\infty x^b e^{-ax}dx = \frac{\Gamma(b+1)}{a^{b+1}} \qquad a>0, b>-1$$
$$= \frac{b!}{a^{b+1}} \qquad a>0, b=0,1,2,3,\dots$$

found in table of integrals of type $[\mathbf{1}]$***, were given. Some of these integrals can also be evaluated by the longer method of integration by parts.***

1.0 Introduction

1.1 Laplace Transform

1.1.1 Definition

The laplace transform of a function $f(x)$ is defined by the integral

$$L\{f(x)\} = \int_0^\infty e^{-x} f(x)dx = \hat{f}(s) \qquad (1.1.2)$$

Where $s \geq 0$ and $0 \leq x \leq \infty$

1.2 Conditions For A Function To Be Laplace Transformable

Not all functions have Laplace transform. A function $f(x)$ has a Laplace transform of $f(x)$ is defined and piecewise continuous on every finite interval on the semi-axis $x \geq 0$ and

$$\big(f(x)\big) \leq Me^{kx} \qquad (1.2.1)$$

for all $x \geq 0$ and some constants M and $K.$ The Laplace transform $L\{f(x)\}$ then exist for all $s > k.$

1.3 Improper Integrals

The integral

$$I\,[a,b] = \int_a^b f(x)dx \qquad (1.3.1)$$

Is called an improper integral if $a = -\infty$ or $b = \infty$ or $both$ i.e. one or both integration limit is infite,
$f(x)$is unbounded at one or more points of $a \leq x \leq b$ such point are called singularities
of$f(x)$

Integrals corresponding to 1.3.1 are called improper integrals of the first and second kinds respectively. Integrals with both condition (1.3.1) and (1.3.2) are called improper integrals of the third kind

Example (1.3.3)

$1 = \int_0^\infty \sin x^2\, dx$ is an integral of the first kind

Example (1.3.4)

$I = \int_0^4 \frac{dx}{x-3}$is an improper integral of the second kind

Example (1.3.5)

$I = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx$ is an improper integral of the third kind

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1.4 Improper Integrals Of The Third Kind

Definition 1.4.1

Let $f(x)$ be bounded and integrable in every finite interval $a \geq x \leq b$, then we define

$$I = \int_0^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx \qquad (1.4.2)$$

The integral on the left is called convergent or divergent according as the limit on the right does or does not exist. Similarly, we define

$$\int_{-\infty}^{b} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx \ (1.4.3)$$

and call the integral on the left convergent or divergent according as the limit on the right does or does not exist.

Example (1.4.4)

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b \to \infty} \int_a^b \frac{dx}{x^2} = \lim_{b \to \infty} \left[1 - \frac{1}{b}\right] = 1$$

So that, $\int_1^\infty \frac{dx}{x^2}$ converges to 1

Some convergence criteria for improper integrals of the first kind

(a) Convergence

let$g(x) \geq 0 \ for\ all\ x \geq a$, and suppose that

$\int_a^\infty g(x)dx$converges then if

$$0 \leq f(x0 \ \leq g(x0 \ for\ all\ x \geq a$$

$$\int_a^\infty f(x)\, dx \ also\ converges$$

(b) Divergence

Let $g(x)\, dx \ \geq 0 \ for\ all\ x \geq a$, and suppose

$$\int_a^\infty g(x)\, dx \ \ diverges\ then\ if$$

$$f(x) \geq g(x) for\ x \geq a$$

$$\int_a^\infty f(x)\, dx \ also\ diverges$$

Theorem 1.4.5 (special comparison Test)

Let $\lim_{x \to \infty} x^p f(x) = A,$

Then

$$\int_a^\infty f(x)\, dx \ \ converges\ if\ P \geq 1\ and\ A \neq \infty \qquad (1.4.6)$$

$$\int_a^\infty f(x)\, dx \ \ converges\ if\ P \leq 1\ and\ A \neq 0 \qquad (1.4.7)$$

(A may be) infinite hence the improper integral is of first kind and theorem 1.4.5 can be applied. Now

$$\lim_{x \to \infty} \frac{x\, Inx}{x+a} = \lim_{x \to \infty} Inx \Big/ (x+a)\big/_x = \lim_{x \to \infty} \frac{Inx}{1 + \frac{a}{x}} = \frac{\infty}{1+0}$$

Hence by theorem 1.4.5, the integral diverges, since $P = 1\ and\ A = \infty$

1.5 Theorem

The Laplace transform is an improper integral of the first kind

Proof

By definition (1.1.2) and (1.2.1) Laplace transform of $f(x)$ is defined by

$$L\{f(x)\} = \int_0^\infty e^{-sx}\ f(x)dx \qquad (1.5.1)$$

And exists if

$$|f(x)| \leq Me^{kx} \qquad s > k$$

i.e. $-Me^{kx} \ \leq f(x) \ \leq \ Me^{kx}$ (1.5.2)

By definition (1.4.1) and theorem (1.4.5), the improper integral $\int_0^\infty e^{-xs}\ f(x)dx$ will converge if

$$\lim_{x \to \infty} x^P e^{-xs} f(x) = A\ and\ P \ \geq, A \neq \infty$$

Considering $\frac{x^p}{e^{-xs}}\ f(x)\ and\ the\ fact\ from$ (1.5.2), we see that, for the inequality on the right hand side

$$\lim_{x\to\infty}\frac{x^P}{e^{xs}}f(x)\ \le \lim_{x\to\infty}\frac{x^P M e^{kx}}{e^{xs}} \quad = M\lim_{x\to\infty}\frac{x^P}{s^{(s-x)x}} \qquad s>k$$

Now for all $p>1$

$$\lim_{x\to\infty}\frac{x^P}{e^{(s-k)x}}=0 \qquad (because\ e^{(s-k)x}\ get\ to\ \infty\ faster\ than\ x^p)$$

Hence $\lim_{x\to\infty} x^p e^{-xs}\,f(x)\le 0$

For the inequality on the left hand side, we see that

$$f(x)\ge\ -Me^{ks},\qquad s>k$$

Hence $\quad \frac{x^p}{e^{xs}}\ f(x)\ge\ -Me^{ks}\frac{x^p}{e^{xs}}\ f(x)$

So that $\quad \lim_{x\to\infty}\frac{x^p}{e^{xs}}\ f(x)\ \ge\ -M\lim_{x\to\infty}\frac{x^P}{e^{(s-x)x}}=0(S>k)$

Consequently

$$\lim_{x\to\infty} x^p e^{-sx}\,f(x)=0$$

Therefore $\int_0^\infty e^{-sx}\,f(x)dx\ \ converges\ for\ p\ \ge\ 1$

Hence the result.

2.0 Evaluation Of Improper Integrals By Methods Of Laplace Transform

Our goal is to show that the following improper integrals are of the first kind and can be evaluated by method of Laplace transform

(a) $\int_0^\infty x^b e^{-ax}\,dx\ \ a>0,\quad (i)\ b>-1\quad\ (ii) b=0,1,2,3\ ...$
(b) $\int_0^\infty x\ e^{-ax}\sin bx\,dx$
(c) $\int_0^\infty e^{-ax}\ bx\ \ dx$
(d) $\int_0^\infty x^2e^{-ax}\sin bx\,dx$
(e) $\int_0^\infty x^2e^{-ax}\cos bx\,dx$

2.1 Obtaining the formulas

$$if\quad b=1,2,3,\dots,\qquad a>0,\qquad we\ set\ b=n\ (positive\ integers)$$

Then $\quad \int_0^\infty x^b e^{-ax}\,dx\ \ =\ \int_0^\infty x^n e^{-sx}\,dx\ \ = L\,\{x^n\},\qquad\qquad s=a$

Which is one of the property of Laplace transform given in [3]

Hence

$$\int_0^\infty x^b e^{-ax}\,dx\ \ =\ \frac{n!}{a^{n+1}}=\ \frac{b!}{a^{b+1}}(b=n) \tag{2.1.1}$$

(a) (i) for this case, i.e. $a\ >0\ and\ b>-1$ we need the definition of the gamma function given as [1, 2]

$$\Gamma(Z)\ \ =\ \ \int_0^\infty e^{-x,}x^{-z-1}\,dx \tag{2.1.2}$$

Now let $ax=u, then\ x=0\ gives\ u=0\ and\ x=\ \infty\ yields$

$$u=\infty\quad\ while\quad\ dx=\frac{1}{a}\ du$$

Hence

$$\int_0^\infty x^b e^{-ax}\,dx\ \ =\ \int_0^\infty\left(\frac{u}{a}\right)^b e^{-u}\left(\frac{du}{a}\right)=\ a^{\frac{1}{b-1}}\int_0^\infty u^b e^{-u}\ \ du$$

$$=\ \frac{\Gamma(b+1)}{a^{b+1}} \tag{2.1.3}$$

Problem (b), (c), (d), and (e) can be evaluated from the special cases of (2.1.1) that is

$$\int_0^\infty x^n e^{-ax}\,dx=\frac{n!}{a^{n+1}}\ \ when\ n=1\ and\ n=2$$

$$case\ 1\qquad\qquad (n=1)$$

$$\int_0^\infty xe^{-ax}\,dx=\frac{1}{a^2}\qquad\qquad a>0 \tag{2.1.4}$$

Therefore (2.1.4)can be used in evaluating the integral for a $>b$

$$\int_0^\infty xe^{-ax}\,dx=\ \int_0^\infty xe^{-(a-b)x}\,dx=\frac{1}{(a-b)^2} \tag{2.1.5}$$

in (2.1.5)replace b by ib to get

$$\int_0^\infty xe^{-ax}e^{ibx}\,dx = \frac{1}{(a-ib)^2}$$

That is

$$\int_0^\infty xe^{-ax}(\cos bx + isin\,bx)\,dx = \frac{1}{a^2-b^2-2iab}\ or$$

$$\int_0^\infty xe^{-ax}\cos bx\ \ dx + i\int_0^\infty xe^{-ax}\sin\ bx\ \ dx$$

$$= \frac{a^2-\ b^2+\ 2iab}{(a^2-b^2-2aib)(a^2-b^2+2aib)} =$$

$$\frac{a^2-\ b^2}{(a^2-b^2)^2+\ 4a^2b^2} + \ i\ \frac{2ab}{(a^2-b^2)^2+\ 4a^2b^2}$$

Hence

$$\int_0^\infty xe^{-ax}\cos bx\ \ dx\ =\ \frac{a^2-b^2}{(a^2+b^2)^2},\qquad a > b \tag{2.1.6}$$

$$\int_0^\infty xe^{-ax}\sin bx\ \ dx = \frac{2ab}{(a^2+b^2)^2}$$

Equation (2.1.5) and (2.1.6) are (b) and (c0 required. For (d) and (c), we use (2.1.1) with $n = 2$

That is

$$\int_0^\infty xe^{-ax}\ \ dx\ =\ \frac{2}{a^2},\ \ a > 0 \tag{2.1.7}$$

Therefore

$$\int_0^\infty xe^{-ax}e^{bx}\,dx\ =\ \int_0^\infty x^2e^{-(a-b)x}\,dx = \frac{2}{(a-b)^3}a > b$$

Next replace b by ib to get

$$\int_0^\infty x^2e^{-ax}e^{ibx}\,dx\ =\ \frac{2}{(a-b)^3}$$

$$\int_0^\infty x^2e^{-ax}(\cos bx + i\sin bx)dx\ \ =\ \ \frac{2(a+ib)^3}{(a-ib)^3(a-ib)^3}$$

$$\int_0^\infty x^2\ e^{-ax}\cos bx\,dx + i\int_0^\infty x^2e^{-ax}\sin bx\,dx$$

$$= \frac{2\{a^3+\ 3\ a^2(ib)+\ 3a\ (ib)^2+\ (ib)^3\}}{(a^2+b^2)^3} + \ i\ \frac{2(3a^2b-\ b^3)}{(a^2+b^2)^3}$$

Hence

$$\int_0^\infty x^2\ e^{-ax}\cos bx\,dx\ \ =\ \ \frac{2\,a^3-6ab^2}{(a^2+b^2)^3} \tag{2.1.8}$$

And

$$\int_0^\infty x^2\ e^{-ax}\sin bx\,dx\ \ =\ \ \frac{6a^2b-2\,b^3}{(a^2+b^2)^3} \tag{2.1.9}$$

Equations (2.1.8) and (2.1.9) are the values sought for (d) and (e)

3.0 Conclusion

In this paper, we presented the term “improper integral” of the first kind and made a brief explanation of when an integral is said to be improper and then established the way of evaluating some improper integrals with laplace transform technique. We proved that every laplace transform is an improper integral of the first kind. Time constraint confined us to a few integrals involving exponential functions.

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