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Closed form Solution of Some Nonlinear Partial Differential Equations

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Abstract

Method of finding the closed form solution of nonlinear partial differential equations using the Monge method is discussed. The method leads to finding one or two intermediate integrals from which a complete integral which is the solution of the given nonlinear partial differential equation is got by eliminating some arbitrary functions. The method is demonstrated by finding the closed form solution of some typical nonlinear partial differential equations.

Keywords: Measure space, Measurable function, Integral, Radon-Nikodym Theorem.

1.0 Introduction

A search through literature reveals that, it is difficult to obtain closed form solutions for a large class of nonlinear partial differential equations. Nevertheless most real life models result into nonlinear partial differential equations whose solutions are most desired in closed forms. Series of attempts have been made and are continuously being made by numerous authors to provide methods that could lead to solution of nonlinear partial differential equations in closed form. Unfortunately every presentation has its own limits in applications.

In 1991 Rubel [1] suggested a method which can find a closed-form solution for some nonlinear partial differential equations by using a method he described as finding the quasi solutions. It has not been easy to advance this method since it involves, firstly considering an equation of higher order which may be easy to be solved and whose solution must reduce to the given equation on differentiation. Johnson and Swoller [2] presented a method that can only be applied to the hyperbolic equations in 1967. Erumaka [3] expounded the method of reduction into simple waves outlined by Fritz [4] in 1982. Again this method can only be applied to nonlinear partial differential equations of the form

$$f(x, t, u, u_x, u_t)u_{xx} = u_{tt} ,$$

and not all nonlinear partial differential equations can be reduced to the above form. For some other methods, see [5 – 8].

In application, most authors have resorted to using numerical methods with all its unavoidable constraints. Many such numerical schemes abound [9 – 12].

In this paper we show how the method popularly referred to as the Monge method can be used to find the closed form solutions of a larger class of nonlinear partial differential equations. In section 2 the different steps involved in the said method is highlighted and the very crucial ones vividly explained. In section 3 we present the application of the Monge method to finding the closed form solutions for some typical nonlinear partial differential equations that often arise in science and engineering. The examples are nonetheless inexhaustive but serve to highlight the beauty in the application of the method.

2.0 The Monge Method

Given the general partial differential equation in two variables x and y (say) of the form

$$F(x, y, u, p, q, r, s, t) = 0, \tag{2.1}$$

where $p=u_x$, $q=u_y$, $r=u_{xx}$, $s=u_{xy}$, $t=u_{yy}$ and

$$u = u(x, y) \tag{2.2}$$

The method consists of establishing one or two intermediate integrals of the form

$$\zeta = f(\xi), \tag{2.3}$$

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where

$$\zeta = \zeta(x, y, u, p, q),$$

$\xi = \xi(x, y, u, p, q)$,

and f is an arbitrary function such that equation (2.1) can be derived from equation (2.3) when the arbitrary function is eliminated. In [13,14] it has been established that for equation (2.1) to possess a first integral of the form (2.3) it must be expressible in the form

$$Rr + Ss + Tt + U(rt - s^2) = V, \qquad (2.4)$$

where R, S, T, U and V are functions x, y, u, p and q.

In this paper we consider the special case in which U = 0 so that equation (2.4) reduces to

$$Rr + Ss + Tt = V \qquad (2.5)$$

Now, by successively differentiating (2.2) we obtain the two relations

$$dp = rdx + sdy \qquad (2.6)$$

$$dq = sdx + t\,dy \qquad (2.7)$$

From (2.6) and 92.7) we obtain

$$r = \frac{dp - sdy}{dx} \qquad (2.8)$$

$$t = \frac{dp - sdy}{dy} \qquad (2.9)$$

Substituting (2.8) and (2.9) into (2.5) we obtain, on simplification,

$$R\,dpdy + T\,dqdx - V\,dxdy = s\,[R\,dy^2 - S\,dxdy + T\,dx^2] \qquad (2.10)$$

It has also been established in [13] that to realize the first intermediate integrals we must have from (2.10) the two conditions

$$R\,dpdy + T\,dq\,dx - V\,dx\,dy = 0 \qquad (2.11)$$

and

$$R\,dy^2 + T\,dx^2 - S\,dxdy = 0 \qquad (2.12)$$

Equations (2.11) and (2.12) are called Monge's subsidiary equations.

A careful inspection of equation (2.12) shows that it is, in principle, possible to find two functions m_1 and m_2 such that equation (2.12) can be expressed as a product of two factors

$$(dy - m_1\,dx)(dy - m_2 dx) = 0 \qquad (2.13)$$

If m_1 and m_2 are distinct, we arrive at the conditions

$$dy - m_1\,dx = 0 \qquad (2.14)$$

and

$$dy - m_2\,dx = 0 \qquad (2.15)$$

By using (2.14) and (2.15) in (2.11) in turn, together with

$$du = pdx + qdy \qquad (2.16)$$

we obtain the intermediate integrals of the form

$$u_1 = f(v_1)\,, \qquad (2.17)$$

$$u_2 = f(v_2)\,, \qquad (2.18)$$

and between (2.17) and (2.18) we obtain p and q in terms of x, y, and u. Substituting these expressions for p and q into (2.16) and integrating, we obtain the required complete integral which gives the general solution of (2.1).

The solution is a bit simplified if $m_1 = m_2$ in (2.14) and (2.15). In this case, substitution into (2.11) and using (2.16) results into one intermediate integral of the form

$$ap + bq = c \qquad (2.19)$$

Integrating (2.19) for u results to the required complete integral

3.0 Illustrative Examples

In this section we illustrate the use of the Monge method so far described to solve nonlinear partial differentials equations in the above category.

Example 1. Solve the equation

$$q^2r + (q - 2\,pq)\,s + (p^2 - p)\,t = 0. \qquad (3.1)$$

Compared with (2.5) we have

$$R = q^2,\ S = q - 2\,pq,\ T = p^2 - p,\ V = 0 \qquad (3.2)$$

Substituting (3.2) into (2.11) and (2.12) we obtain

$$q^2\,dpdy + (p^2 - p)dq\,dx = 0 \qquad (3.3)$$

$$q^2\,dy^2 - (q - 2pq)\,dx\,dy + (p^2 - p)\,dx^2 = 0 \qquad (3.4)$$

It is easy to see that (3.4) can be written as a product of two factors in the form

$$(p\,dx + q\,dy)\,(pdx + qdy - dx) = 0 \qquad (3.5)$$

Hence we have the two conditions

$$pdx + q\,dy = 0 \qquad (3.6)$$

and

$$pdx + qdy - dx = 0 \tag{3.7}$$

Using equations (3.6) and (3.7) in (3.3) we obtain the systems

$$\begin{cases} q^2 dpdy + p(p-1)dq \, dx = 0 \\ p \, dx + qdy = 0 \end{cases} \tag{3.8}$$

and

$$\begin{cases} q^2 dpdy + p(p-1)dq \, dx = 0 \\ p \, dx + qdy - dx = 0 \,, \end{cases} \tag{3.9}$$

respectively.

Solving (3.8) and (3.9) in turn we obtain the first intermediate integrals

$$p - q\, f(u) = 1 \tag{3.10}$$

and $$p - q\, f(u - x) = 0 \tag{3.11}$$

respectively

Eliminating p and q between (3.10) and (3.11) we obtain the relation

$$f(u - x) = f(u)\, du + dy \tag{3.12}$$

Integrating (3.12) we obtain the general solution of (3.1) as

$$y + g(u) = h\,(u - x) \tag{3.13}$$

Example 2. Solve the equation

$$(r - s)y + (s - t)x + q - p = 0. \tag{3.14}$$

Compared with (2.5) we have

$$R = y\,, \quad S = x - y, \qquad T = -x, \quad V = p - q \tag{3.15}$$

Substituting (3.15) into the Monge's subsidiary equations we obtain

$$y\, dp\, dy = xdqdx - (p - q)\, dx\, dy) = 0 \tag{3.16}$$

$$y\, dy^2 - (x - y)dxdy - xdx^2 = 0 \tag{3.17}$$

Equation (13.17) factors into

$$(ydy - xdx)(dy + dx) = 0, \tag{3.18}$$

which implies that

$$ydy - x\, dx = 0 \tag{3.19}$$

$$dy = -\, dy \tag{3.20}$$

Noting that from (3.20) we have

$$y + x = c_1 \tag{3.21}$$

and substituting (3.19) in (3.16), we obtain after simplification

$$d(yp) + d(x\, q) = 0 \tag{3.22}$$

Integrating (3.22) and using (3.21) we obtain the general solution of (3.14) as

$$u - F(v) = a_2, \tag{3.23}$$

where

$$v = x + \sqrt{x^2 + a} \tag{3.24}$$

Example 3. Solve the equation

$$r + (a + b)s + abt = xy \tag{3.25}$$

Compared with (2.5) we have

$$R = 1,\ S = (a + b),\ T = ab,\ V = xy \tag{3.26}$$

Substituting (3.26) into (2.11) and (2.12) we obtain

$$dp\, dy + abdqdx - xy\, dx\, dy = 0 \tag{3.27}$$

$$dy^2 - (a + b)\, dxdy + ab\, dx^2 = 0 \tag{3.28}$$

Equation (3.28) factors into

$$(dy - bdx)\, (dy - adx) = 0 \tag{3.29}$$

Equation (3.29) and (3.27) lead to the systems

$$\begin{cases} dp\ dy + ab\ dq\ dx - xy\ dx\ dy = 0 \\ dy - bdx = 0 \end{cases} \tag{3.30}$$

and

$$\begin{cases} dp\ dy + ab\ dq\ dx - xy\ dx\ dy = 0 \\ dy - adx = 0 \end{cases} \tag{3.31}$$

From (3.30) we realize the first intermediate integral as

$$p + aq = \frac{x^2 y}{2} - \frac{bx^3}{6} + f_1(y - bx)\,, \tag{3.32}$$

where f_1 is arbitrary

Again, from (3.31) we obtain the second intermediate integral as

$$p + bq = \frac{x^2 y}{2} - \frac{bx^3}{6} + f_2(y - ax) \tag{3.33}$$

where f_2 is arbitrary.

Eliminating p and q between equations (3.32) and (3.33) and integrating the result we obtain the complete integral of (3.25) as

$$u = \frac{1}{6}x^3y - \frac{1}{24}(a + b)x^4 + \psi_1(y - bx) + \psi_2(y - ax), \tag{3.34}$$

where ψ_1 and ψ_2 are arbitrary functions.

4.0 Summary and Conclusion

We have outlined the Monge method for solving the nonlinear partial differential equations of the form (2.5), explaining the different steps involved in getting a closed form solution for such equations. The examples presented in section 3 showed clearly how this method can readily be applied to determine the closed form solution of (2.5). Although, not all nonlinear partial differential equations are of the form (2.4) or (2.5) the work of [13] shows how a large class of nonlinear partial differential equations can be transformed into the form (2.4) and (2.5) given some established conditions. In this paper which serves as a first part on this discussion we have considered the general case, (2.4) in which U = 0 reducing it to (2.5). In the second part to be presented we shall present the method for the general case of (2.4)

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