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A Remark on the Integrability of the Sum and Difference of Real-Valued Functions

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Abstract

We establish a theorem on the integrability of the sum $f + g$ and the difference $f - g$ of extended real-valued integrable $f, g : (X, A, \mu) \to \mathbb{R}^e = \mathbb{R} \cup \{\infty, -\infty\}$ ((X, A, μ) a measure space) that is implicit in the literature and employed but rarely explicitly stated (if at all) and rarely explicitly proved. We point out the important instance of the proof of Fubini's Theorem where it is employed by a number of authors, but without citing it.

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1.0 Introduction

Our language and notation shall be pretty standard as found for example, in [1], [2], [3] and [4]. We denote by $f\,|D$ the restriction of the function $f : A \to B$ to the subset $\varnothing \neq D$ of A. $\mathbb{R}$ denotes the real numbers and $\mathbb{R}^e = \mathbb{R} \cup \{\infty, -\infty\}$. /// signifies the end or absence of a proof.

THROUGHOUT, (X, A, μ) is a measure space.

We state straightaway the theorem in question.

THEOREM 1 Let (X, A, μ) be a measure space and $f, g : X \to \mathbb{R}^e$ extended real-valued integrable functions. Then,

(i) the function

$$h : X \to \mathbb{R}^e = \mathbb{R} \cup \{\infty, -\infty\}$$
$$x \mapsto \begin{cases} 0, & \text{if } f(x) + g(x) = \infty - \infty \text{ or } -\infty + \infty \\ f(x) + g(x), & \text{otherwise,} \end{cases}$$

is integrable, and

(ii) $$\int_X h \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu. \qquad \ldots(\Delta)$$

[| **Note:** The minus sign – can replace + in $f(x) + g(x)$ with (Δ) then becoming $\int_X h \, d\mu = \int_X f \, d\mu - \int_X g \, d\mu$ |] . Simply consider f and $-g$.

REMARK 2(a) The measurability of h in the claim (i) is as given in Exercise 2.1.2(b), p.46 of [5], with α there set = 0 here. See COROLLARY 7 below.

(b) Compare Exercise 2.3.5, p.69 of [1] : Let (X, A, μ) be a measure space, let f, $g : X \to [-\infty, \infty]$ be integrable, and let $h : X \to [-\infty, \infty]$ be an A -measurable function that satisfies

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$h(x) = f(x) + g(x)$ at μ-almost every x in X. Show that h is integrable and that $\int_X h \, d\mu = \int_X f \, d\mu + \int_A g \, d\mu$.

2.0 Proof of Theorem 1

We fist establish the measurability of the function h through a sequence of what we call *lemmas*. Throughout, of course, (X, A, μ) is a measure space, and $f, \ g : X \to \mathbb{R}^e$ extended real-valued functions.

LEMMA 3 A constant extended real-valued function $f : X \to \mathbb{R}^e$ is measurable **.** ///

LEMMA 4 (See Theorem 5.5(iii), p.116 and the two lines preceding Theorem 5.3, p.105 of [2]… In the following theorem therefore, we assume that the functions are such that the algebraic operations are possible and Theorem VI.1.6, p.83 of [3]). Let $f, g : X : \to \mathbb{R}^e$ be extended real-valued measurable functions such that $f + g \,/\, f - g$ is *everywhere* defined. Then, $f + g \,/\, f - g$ is also measurable. ///

LEMMA 5 The restriction $f\,|D$ of extended real-valued measurable $f : X \to \mathbb{R}^e$, to a measurable subset $\varnothing \neq D$ of X is measurable.

Proof For $\alpha \in \mathbb{R}$, $(f\,|D)^{-1}((\alpha, \infty]) = D \cap f^{-1}((\alpha, \infty])$. ///

LEMMA 6 (See Problem 3.21(a), p.69 of [6] and Exercise 2.1.5, p.47 of [5]). Suppose $\varnothing \neq E, D \in A$. Then, $f : D \cup E \to \mathbb{R}^e$ is an extended real valued measurable function $\Leftrightarrow$ $f\,|D$ and $f\,|\,E$ are measurable functions.

Proof If $D \subseteq E$ or $E \subseteq D$, we have nothing to show by LEMMA 5. So, suppose $D \nsubseteq E$ and $E \nsubseteq D$. So, as in Fig. 1 below, neither D nor E includes the other.

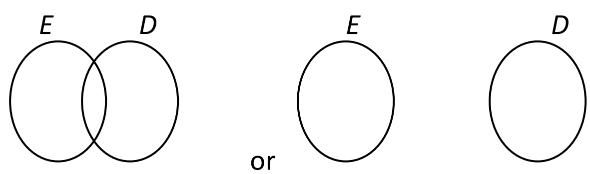

Fig.1 E, D, neither containing the other.

The implication $\Rightarrow$ is immediate from LAMMA 5. For the implication $\Leftarrow$ assume that $f\,|D$ and $f\,|\,E$ are measurable functions. Then, By LEMMA 5, $f\,|D - E$, $f\,|\,E - D$ and (provided $D \cap E \neq \varnothing$) $f\,|D \cap E$ are measurable functions. Let $\alpha \in \mathbb{R}$. Then,

$\{x \in D - E : f(x) > \alpha\}$, $\{x \in E - D : f(x) > \alpha\}$ and $\{x \in D \cap E : f(x) > \alpha\}$

are measurable sets. And so their union is measurable. Clearly, since $(D - E) \cup (D \cap E) \cup (E - D)$ is a disjoint union, then the measurable disjoint union

$$\begin{aligned}
&\{x \in D - E : f(x) > \alpha\} \cup \{x \in D \cap E : f(x) > \alpha\} \cup \{x \in E - D : f(x) > \alpha\} \\
&= \{x \in (D - E) \cup (D \cap E) \cup (E - D) : f(x) > \alpha\} = \{x \in D \cup E : f(x) > \alpha\} \\
&= f^{-1}((\alpha, \infty]).
\end{aligned}$$

Since α was arbitrary, f is measurable. ///

COROLLARY 7 (See Exercise 2.1.2(b), p.46 of [5]and Problem 3.22(c), p. 69 of [6]) Let $f, g : X \to \mathbb{R}^e$ be extended real-valued measurable functions. Let $\alpha \in \mathbb{R}^e$, and define

$$h : X \to \mathbb{R}^e,$$

$$z \mapsto \begin{cases} \alpha, & \text{if } z \in \{x \in X : f(x) = \infty\} \cap \{x \in X : g(x) = -\infty\} \\ \alpha, & \text{if } z \in \{x \in X : f(x) = -\infty\} \cap \{x \in X : g(x) = \infty\} \\ f(z) + g(z), & \text{otherwise.} \end{cases}$$

Then, h is measurable.

Proof Let

$A = \{x \in X : f(x) = \infty\} \cap \{x \in X : g(x) = -\infty\}$, and

$B = \{x \in X : f(x) = -\infty\} \cap \{x \in X : g(x) = \infty\}$.

Since f and g are measurable functions, A and B are measurable sets, and so $A \cup B$ is a measurable set. If $A \cup B = X$, then h is a constant function and, by LEMMA 3, measura- ble; and we have finished the proof. So, suppose $A \cup B \subsetneq X$ and so $C = X - (A \cup B) \neq \varnothing$. Hence, $C = X - (A \cup B)$ is a non-empty measurable set. Clearly, then,

$$X = C \cup (A \cup B) \qquad \ldots(\rho)$$

By LEMMA 5, $f|C$ and $g|C$ are measurable functions, and so by LEMMA 4, $f|C + g|C$ is a measurable function. Clearly, $h|C = f|C + g|C$ and so

$$h|C \text{ is a measurable function} \qquad \ldots(*)$$

$$\left.\begin{array}{l}\text{Clearly, } h|A \cup B \text{ is a constant function which by LEMMA 3} \\ \text{is a measurable function}\end{array}\right\} \qquad \ldots(**)$$

By (ρ), (*) and (**) and LEMMA 6, h is a measurable function. ///

DIGRESSION 8 We note that the following is an immediate corollary of LEMMA 3 and LEMMA 6.

Corollary 9 Let (X, A, μ) be a measure space and $D \cup E$ a disjoint union of non-empty members D, E of A. Define $f : D \cup E \to \mathbb{R}^e$ and suppose

(i) $f|D$ is measurable, and

(ii) $f|E = 0$(i.e., $f|E$ is the zero function on E).

Then, the function f is a measurable function. ///

This corollary is widely used in the literature.

Example 10 The claims in the proof of Theorem 3.4.1, p. 106/107 of [1], that the two functions f in that proof are measurable, is an application of the above corollary.

Example 11 And the measurability of the function

$$\sum_1^\infty f_j = \begin{cases} \sum_1^\infty f_j(x), & \text{if } \sum_1^\infty |f_j(x)| < \infty \\ 0, & \text{if } \sum_1^\infty |f_j(x)| = \infty \end{cases}$$

of the proof of Theorem 2.25, p.53 of [2] comes from an application of this corollary.///

Example 12 (See Example in line 15 on page 156of [4]) The function

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto \begin{cases} \frac{1}{\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

is $\mathcal{B}(\mathbb{R})$-measurable where, of course, $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel sets of $\mathbb{R}$, and the restriction $f|(0,1)$ of f to the open interval $(0, 1)$ is a continuous function. Of course, the union $(-\infty, 0] \cup [1, \infty)$ of two (infinite) intervals $\in \mathcal{B}(\mathbb{R})$, since $\mathcal{B}(\mathbb{R})$ is a σ-algebra and so contains countable unions of its members. ///

This is another instance of a theorem widely used in the literature but which is rarely stated (if at all) and whose proof is rarely explicitly given. End of digression.

Corollary 7 proves the measurability of the function h in our THEOREM 1.We conclude its proof by now showing that h is integrable and that (Δ) in the theorem is true.

Proof For ease of reference, we state a

LEMMA 13(See Theorem 5.5(iii), p.116 of [2] and Lemma 4). Suppose that extended real-valued $f, g : X \to \mathbb{R}^e$ are integrable, and that $f + g$ is everywhere defined. Then, $f + g$ is integrable and $\int_X (f+g)d\mu = \int_X f\, d\mu + \int_X g\, d\mu$. ///

Now let $B = \{x \in X : f(x) = \infty\}$ and $g(x) = -\infty$, or, $f(x) = -\infty$ and $g(x) = \infty\}$. If B

$= \emptyset$ we have noting to show as our theorem then reduces to LEMMA 13 and so suppose $B \neq \emptyset$. Since f and g are measurable functions, $B \in A$. Since f and g are integrable and so finite almost everywhere by Corollary 2.3.12, p.68of [1], $\mu(B) = 0$. Now define the function

$$f^* = f\chi_{B^c}$$

where $B^c = X - B$. Then,

f^* is measurable

and $f = f^*$ almost everywhere.

And so by Proposition 2.3.8, p.67 of [1] and the hypothesis that f is integrable, f^* is integrable with same integral, i.e.,

$$\int_X f\, d\mu = \int_X f^*\, d\mu \qquad \ldots(\nabla)$$

Similarly, define $g^* = g\chi_{B^c}$, and clearly,

$$h = f^* + g^*$$

and is everywhere defined. By LEMMA 13 therefore h is integrable and

$$\int_X h\, d\mu = \int_X (f^* + g^*)\, d\mu = \int_X f^*\, d\mu + \int_X g^*\, d\mu$$

which by (∇)

$= \int_X f\, d\mu + \int_X g\, d\mu$. ///

3.0 An Important Instance

The statement "consequently I_f belongs to $\mathcal{L}^1(X, A, \mu)$" in p.160, lines 9-10 of [1] of the proof of Theorem 5.2.2 [Fubini's Theorem] is an instance of application of our THEOREM 1. The same application is recorded in the phrase "Hence $\varphi \in L^1(\mu)$" in line 18 of page 152 of [4].

References

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