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A Remark on the Definition of the Integral

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Abstract

Let (X, A, μ) be a measure space, $\emptyset \neq A \in$ A, $A \neq X$, and $f : X \to \mathbb{R}^e$ and $g : A \to \mathbb{R}^e$ extended real-valued measurable functions. The literature defines the integral of f [| the ground set of the measure space (X, A, μ), X, is the domain of f |] directly, but defines the integral of g [| the domain of g is not the whole of X |](See "(rather than the entire space)" line 10, p.65 of [1]) as the integral of $g^* : X \to \mathbb{R}^e$ (if it exists)(See "(if it exists)" line 11, p.65 of [1])

where $g^*(x) = \begin{cases} g(x), & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$

In this paper we define the integral of measurable $h : A \to \mathbb{R}^e$, $A \in$ A, $A \neq X$, or A = X, directly. Our definition, which is essentially the literature's definition, captures the definition of the integrals of f and g in the literature. Some useful deductions and consequences are then given.

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1.0 Introduction

Our language and notation shall be standard, as found for example in [1] and [2]. We denote by $f\,|D$ the restriction of the function $f : C \to B$ to $\emptyset \neq D \subseteq C$. Without mention we work *throughout* with the measure space (X, A , μ), and, of course(See Proposition 2.1.1, p.48 of [1] and Proposition 3.5.18, p.65 of [3]), the domain A of a measurable function $f : A \to \mathbb{R}^e = \mathbb{R} \cup \{\infty, -\infty\}$, $\mathbb{R}$ = the reals, belongs to the σ-algebra A. We signify the end or absence of a proof by ///.

THROUGHOUT, (X, A, μ) is a fixed measure space, and $\emptyset \neq A \in$ A. With f, g, g^* as in the abstract

(i) The literature's definition of the integral of g now becomes a theorem : lit's $\int_A g\, d\mu \equiv \int_X g^*\, d\mu =$ new $\int_A g\ d\mu$. [| Section 4's THEOREM 4.9|].
(ii) Thus, [| Section 4's Corollary 4.11 and Example 4.12|], we show that the claim of Exercise 1, p.188 of [4] is true in general and not just for the Lebesgue integral of $f : [a, b] \to \mathbb{R}$.
(iii) Similarly, for the *integral of f over A*,
old $\int_A f\, d\mu \equiv \int_X f\chi_A\ d\mu =$ new $\int_A f\,|\,A\ d\mu$.
(iv) Very importantly, some new and clearer proofs of elementary properties of the integral are also obtained.
(v) Finally, we are able to explain *satisfactorily* (COMMENT 4.13) the reasoning in the last sentence of the first paragraph of page 134 of [2] in its proof of the Radon-Nikodym Theorem.

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2.0 Measurable Simple Function

Definition of a Simple Function 1.1 Let $\varnothing \neq A \subseteq X$. The real-valued function $s : A \to \mathbb{R}$ shall be called a *simple function* if it has a finite range $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathbb{R}$, say(See line 15 of p.45 of [5]:*We do not allow simple functions to assume the values* $\pm\infty$), and line 3 of Section VII.1 p.105 of [2] : $\phi(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x)$, a_k real).And so if $A_k = s^{-1}(\alpha_k)$, $k = 1, 2, \ldots, n$, we can write s as a linear combination

$$s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots + \alpha_n\chi_{A_n} \qquad \ldots(\Delta)$$

of characteristic functions, and, following p.47 of [5] and p.78 of [6], call (Δ) the *standard representation* or *canonical representation*(See p.90 of [7]) *of s*. If s is a measurable function, and so $A \in \mathcal{A}$, we call it a *measurable simple function*.

Note 1.2 (i) Our definition of a measurable simple function capture's the literature's definition with $A = X$. However, see the statement of Proposition 2.1.7, page 54 of [1].

(ii) It is clearly true, as one verifies trivially, that our new simple function $s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots + \alpha_n\chi_{A_n}$ is also measurable if and only if each $A_k \in \mathcal{A}$, $k = 1, 2, \ldots, n$.

(iii) $\{A_1, A_2, \ldots, A_n\}$ is a partition of A.[Note: The sets constituting a *partition* are non-empty sets. So, $\varnothing \neq A_k$ for each $k = 1, 2, \ldots, n$.]

We here introduce the notion of an **Acceptable Representation 1.3**(See last two lines of p. 78 of [6]) Let $\varnothing \neq A \subseteq \mathcal{A}$. We shall call the subfamily

$$\{B_1, B_2, \ldots, B_r\} \qquad \text{(ARF)}$$

of $\mathcal{A}$ a *measurable partition* of A if it is a partition of A. If for some $\beta_1, \beta_2, \ldots, \beta_r \in \mathbb{R}$, the measurable simple function $s : A \to \mathbb{R}$ can be represented as

$$s = \beta_1\chi_{B_1} + \beta_2\chi_{B_2} + \ldots + \beta_r\chi_{B_r} \qquad \text{(AR)}$$

we shall call (AR) an *acceptable representation* of s and call (ARF) an *acceptable representing family* for s. [| Compare first Paragraph of Section VII.1 of [2] : …where the $A_k's$ form a measurable pairwise disjoint partition of X. |]

Note 1.4 (i) The family $\{A_1, A_2, \ldots, A_n\}$, $A_k = s^{-1}(\alpha_k)$, $k = 1, 2, \ldots, n$, in the standard representation of measurable simple function $s : A \to \mathbb{R}$ is clearly an acceptable representing family for s.

(ii) The standard representation of measurable simple function $s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots + \alpha_n\chi_{A_n}$ is clearly an acceptable representation of s.

A simple easily digestible proof of the following fundamental theorem is not easy to locate in the literature.

FUNDAMENTAL THEOREM 1.5 If

$$s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots + \alpha_n\chi_{A_n} \qquad (\Delta')$$

and

$$s = \beta_1\chi_{B_1} + \beta_2\chi_{B_2} + \ldots + \beta_r\chi_{B_r} \qquad (\Delta\Delta')$$

are, respectively, *the standard representation* and *an acceptable representation* of the measurable simple function $s : A \to \mathbb{R}$, then,

$$\sum_{k=1}^{n} \alpha_k \mu(A_k) = \sum_{j=1}^{r} \beta_j \mu(B_j) .$$

Proof First observe that each β_j is equal to one and only one α_k since (Δ') is the standard representation. We note next that, clearly, then

(i) for each $j \in \{1, 2, \ldots, r\}$, $B_j \subseteq A_k$ for some k with $\beta_j = \alpha_k$,
(ii) for fixed k, there is at least one j such that $B_j \subseteq A_k$, and
(iii) those B_j's contained in A_k, say, constitute a partition of A_k.
So, if

$B_{11}, B_{12}, \ldots, B_1p_1$ are the B_j's contained in A_1,
$B_{21}, B_{22}, \ldots, B_2p_2$ are the B_j's contained in A_2,
.
.
$B_{n1}, B_{n2}, \ldots, B_{np_n}$ are the B_j's contained in A_n,

and so

$\{B_{11}, B_{12}, \ldots, B_{1p_1}\}$ is a partition of A_1,

$\{B_{21}, B_{22}, \ldots, B_{2p_2}\}$ is a partition of A_2,

.
.
.

$\{B_{n1}, B_{n2}, \ldots, B_{np_n}\}$ is a partition of A_n,

then, we have the disjoint unions

$$A_1 = \bigcup_{t=1}^{p_1} B_{1t}, \ A_2 = \bigcup_{t=1}^{p_2} B_{2t}, \ \ldots, A_n = \bigcup_{t=1}^{p_n} B_{nt},$$

$\beta_{1t} = \alpha_1$, for $t = 1, 2, \ldots, p_1$, $\beta_{2t} = \alpha_2$, for $t = 1, 2, \ldots, p_2$, …, $\beta_{nt} = \alpha_n$ for $t = 1, 2, \ldots, p_n$.
By additivity of μ,

$$\alpha_1\mu(A_1) = \alpha_1\mu\left(\bigcup_{t=1}^{p_1} B_{1t}\right) = \alpha_1 \sum_{t=1}^{p_1}\mu(B_{1t}) = \sum_{t=1}^{p_1}\alpha_1\,\mu(B_{1t}) = \sum_{t=1}^{p_1}\beta_{1t}\,\mu(B_{1t}),$$

$$\alpha_2\mu(A_2) = \alpha_2\mu\left(\bigcup_{t=1}^{p_2} B_{2t}\right) = \alpha_2 \sum_{t=1}^{p_2}\mu(B_{2t}) = \sum_{t=1}^{p_2}\alpha_2\mu(B_{2t}) = \sum_{t=1}^{p_2}\beta_{2t}\mu(B_{2t}),$$

$$\alpha_n\mu(A_n) = \alpha_n\mu\left(\bigcup_{t=1}^{p_n} B_{nt}\right) = \alpha_n \sum_{t=1}^{p_n}\mu(B_{nt}) = \sum_{t=1}^{p_n}\alpha_n\mu(B_{nt}) = \sum_{t=1}^{p_n}\beta_{nt}\mu(B_{nt}).$$

Hence

$$\sum_{t=1}^{n}\alpha_k\mu(A_k) = \alpha_1\mu(A_1) + \alpha_2\mu(A_2) + \ldots + \alpha_n\mu(A_n) = \sum_{t=1}^{p_1}\beta_{1t}\mu(B_{1t}) + \sum_{t=1}^{p_2}\beta_{2t}\mu(B_{2t}) +$$

$$\ldots + \sum_{t=1}^{p_n}\beta_{nt}\mu(B_{nt})$$

which clearly, since

$$\{B_{11}, B_{12}, \ldots, B_{1p_1}, B_{21}, B_{22}, \ldots, B_{2p_2}, \ldots B_{n1}, B_{n2}, \ldots, B_{np_n}\} = \{B_1, B_2, \ldots, B_r\},$$

is equal to $\displaystyle\sum_{j=1}^{r}\beta_j\mu(B_j)$.

That is,

$$\sum_{k=1}^{n}\alpha_k\mu(A_k) = \sum_{j=1}^{r}\beta_j\mu(B_j). \ ///$$

The Fundamental Definition of the Integral of Non-negative Measurable Simple Function 1.6 Let the non-negative measurable simple function $s : A \to \mathbb{R}$ have the standard representation

$$s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots. + \alpha_n\chi_{A_n} = \sum_{k=1}^{n}\alpha_k\,\chi_{A_k}.$$

We **here** (i.e., in this paper) define the *integral* of *s*, denoted $\int_A s \ d\mu$ as

$$\alpha_1\mu(A_1) + \alpha_2\mu(A_2) + \ldots + \alpha_n\mu(A_n) = \sum_{k=1}^{n}\alpha_k\mu(A_k).$$

NOTE 1.7 (i) The usual conventions

$$0 \cdot \infty = 0$$

and $\alpha \cdot \infty = \infty$ for $\alpha \in \mathbb{R}$, $\alpha > 0$

are in vogue in the above definition of the integral. E.g., if $\alpha_2 = 4$ and $\mu(A_2) = \infty$, then $\alpha_2\mu(A_2) = 4 \cdot \infty = \infty$. Similarly, if $\alpha_5 = 0$ and $\mu(A_5) = \infty$, then $\alpha_5\mu(A_5) = 0 \cdot \infty = 0$.

(ii) $\int_A s\, d\mu \in [0, \infty] = [0, \infty)\cup\{\infty\} = \mathbb{R}^{e+}$.

(iii) Clearly, for non-negative measurable simple $s : A \to \mathbb{R}$ with $\mu(A) = 0$, $\int_A s\, d\mu = 0$, by the monotonicity of measure.

(iv) Clearly, our definition of the integral of non-negative measurable simple function $s : A \to \mathbb{R}$ captures the literature's definition of the integral of non-negative measurable simple $s : A \to \mathbb{R}$, when $A = X$. (See p.105 of [2] and p.47/48 of [5]).

(v) We have called the preceding Definition 1.6 *Fundamental* as it sets the stage and dic- tates all ensuing definitions and theorems. E. g. see preceding (iii).

(vi) Before defining next, directly, the definition of the integral $\int_A f\, d\mu$ of extended real-valued non-negative measurable $f : A \to \mathbb{R}^{e+}$, we wade through some elementary results on the integral of non-negative measurable simple $s : A \to \mathbb{R}$.

THEOREM 1.8 Let $\varnothing \neq A \in$ A and $s : A \to \mathbb{R}$ a non-negative measurable simple function.

(i) If

$$s = \sum_{k=1}^{n} \alpha_k \chi_{A_k}$$

and

$$s = \sum_{j=1}^{r} \beta_j \chi_{B_j}$$

are, respectively, the standard representation and an acceptable representation of s, then,

$$\int_A s\, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) = \sum_{j=1}^{r} \beta_j \mu(B_j),$$

(ii) If $s, t : A \to \mathbb{R}$ are non-negative measurable simple functions such that $s = t$ almost everywhere, then $\int_A s\, d\mu = \int_A t\, d\mu$.

(iii) (Monotonicity) If $s, t : A \to \mathbb{R}$ are non-negative measurable simple functions such that $s \leq t$ almost everywhere, then $\int_A s\, d\mu \leq \int_A t\, d\mu$.

(iv) If $s, t : A \to \mathbb{R}$ are non-negative measurable simple functions, $\alpha \in \mathbb{R}$, $\alpha \geq 0$, then $\int_A \alpha s\, d\mu = \alpha \int_A s\, d\mu$ and $\int_A (s+t)\, d\mu = \int_A s\, d\mu + \int_A t\, d\mu$.

Proof (i) is immediate from THEOREM 1.5. Compare the definitions of the integral in first paragraph of Section 2.2, p.47 of [5] and in first paragraph of Section VII.1,p.105of [2]…. Specifically,

$$\phi(x) = \sum_{k=1}^{n} \alpha_k \chi_{A_k}(x),\ \alpha_k \text{ real},\ \alpha_k \geq 0,$$

where the A_k's form a measurable pairwise disjoint partition of X. The integral of ϕ over X w.r.t. μ is denoted by $\int_X \phi(x)\, d\mu(x)$, or plainly by $\int_X \phi\, d\mu$, and it is defined as

$$\int_X \phi\, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k).$$

The usual convention $0\cdot\infty = 0$ is in …] .Observe that while [5] uses the standard representation of s to define the integral, $\int_A s\, d\mu$, of s, [2] uses an acceptable representa- tion of s. So, we have here shown that it is correct to use either representations. Before proving (ii) – (iv) we note the following.

Suppose

$$s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots + \alpha_p\chi_{A_p}$$

$$t = \beta_1\chi_{B_1} + \beta_2\chi_{B_2} + \ldots + \beta_q\chi_{B_q}$$

are the standard representations of *s* and *t*. Clearly the non-empty members of the family

$$\{A_k \cap B_j : k = 1, 2, \ldots, p,\ j = 1, 2, \ldots, q\}$$

constitute a partition of *A*, and so suppose this partition is

$$\{ A_{p_1} \cap B_{q_1},\ A_{p_2} \cap B_{q_2},\ \ldots,\ A_{p_r} \cap B_{q_r} \} \qquad (\nabla)$$

Then, clearly,

$$s = \sum_{k=1}^{r} \alpha_{p_k}\, \chi_{A_{pk} \cap B_{qk}}$$

and

$$t = \sum_{k=1}^{r} \beta_{q_k}\, \chi_{A_{pk} \cap B_{qk}}$$

are acceptable representations of *s* and *t*, respectively, and (∇) a common acceptable representing family for *s* and *t*. By (i) of the present THEOREM 1.8, therefore,

$$\int_A s\, d\mu = \sum_{k=1}^{r} \alpha_{p_k} \mu(A_{p_k} \cap B_{q_k}) \qquad (\Sigma s)$$

and

$$\int_A t\, d\mu = \sum_{k=1}^{r} \beta_{q_k} \mu(A_{p_k} \cap B_{q_k}) \qquad (\Sigma t)$$

A careful use of (∑s) and (∑t) gives the proofs of (ii) – (iv). For (ii), for an instance, the union of those members of the common acceptable representing family (∇), on which $s \neq t$, is a measurable set of μ-measure zero, and so their contributions to (∑s) and (∑t) is zero, while each of the remaining members of (∇) contribute same amount to (∑s) and (∑t) since $s = t$ on each of them; and so (∑t) = (∑s).///

OBSERVATIONS 1.9 Let $\emptyset \neq A \in$ A .

(i) $s = \kappa_0 : A \to \mathbb{R}$, $x \mapsto 0$, for all $x \in A$, the constant zero function on *A*, is a simple measurable function. Clearly, $s = \kappa_0 = 0\chi_A$ and $\int_A s\, d\mu - \int_A 0\chi_A = 0\mu(A) = 0$.

(ii) If $\alpha \in \mathbb{R}$, $\alpha \geq 0$, then $\alpha\chi_A$ is a measurable simple function with integral

$$\int_A \alpha\chi_A\, d\mu = \alpha\mu(A).$$

(iii) If $\emptyset \neq B \subsetneq A$ and $\beta \in \mathbb{R}$, $\beta \geq 0$, the measurable simple function $s = 0\chi_A = \beta\chi_A$ is usually written simply as $\beta\chi_B$. And by the definition of the integral

$$\int_A s\, d\mu = \int_A (0\chi_{A-B} + \beta\chi_B)\, d\mu = 0\mu(A-B) + \beta\mu(B) = \beta\mu(B).$$

Hence, $s = \beta\chi_B$ has the integral $\beta\mu(B)$. That is,

$$\int_A \beta\chi_B = \beta\mu(B).$$

Similarly, if $\{B_1, B_2, \ldots, B_n\}$ is a disjoint family of non-empty subsets of *A*, with $\bigcup_{k=1}^{n} B_k \subsetneq A$, define $A - \bigcup_{k=1}^{n} B_k = B$. Suppose $\beta_1, \beta_2, \ldots, \beta_n \geq 0$. Then, the simple measurable function

$$s = 0\chi_{A-B} + \beta_1\chi_{B_1} + \beta_2\chi_{B_2} + \ldots\ldots + \beta_n\chi_{B_n}$$

is same as

$$\beta_1\chi_{B_1} + \beta_2\chi_{B_2} + \ldots\ldots. + \beta_n\chi_{B_n}.$$

By the definition of the integral,

$$\int_A s\, d\mu = 0\mu(A-B) + \beta_1\mu(B_1) + \beta_2\mu(B_2) + \ldots\ldots + \beta_n\mu(B_n)$$

$= \beta_1\mu(B_1) + \beta_2\mu(B_2) + \ldots\ldots + \beta_n\mu(B_n)$.

That is,

$$\int_A (\beta_1\chi_{B_1} + \beta_2\chi_{B_2} + \ldots\ldots + \beta_n\chi_{B_n})\, d\mu$$
$$= \beta_1\mu(B_1) + \beta_2\mu(B_2) + \ldots\ldots + \beta_n\mu(B_n).$$

(iv) The integral was defined using the *standard representation* of the non-negative measurable simple function s, and characterized by THEOREM 1.8(i) using acceptable representations. The situation in the second part of (iii) where a member of the acceptable representing family (a member on which s assumes the value 0) is not shown, is of common occurrence. When defining the integral, however, cognizance is taken of this absent member of the representing family, its contribution being 0. This shall be done in what follows, in several places as is in the literature, without citation.

Let the non-negative measurable simple function $s : A \to \mathbb{R}$ have the standard or acceptable representation

$$s = \sum_{k=1}^{n} \alpha_k \chi_{A_k} \qquad (\rho)$$

and suppose $\varnothing \neq B \subseteq A$, $B \in$ A. Then, the restriction $s|B : B \to \mathbb{R}$, $(s|B)(b) = s(b)$ for all $b \in B$, of s to B, is also a measurable non-negative simple function with the standard or acceptable representation

$$s|B = \sum_{j=1}^{r} \alpha_j \chi_{A_j \cap B}$$

where $A_1, A_2, \ldots, A_r$ are the A_k's in (ρ) having non-empty intersection with B. Hence, by our definition and (i) of THEOREM 1.8, the integral of $s|B$ is

$$\int_B s|B\, d\mu = \sum_{j=1}^{r} \alpha_j\, \mu(A_j \cap B) \qquad (*)$$

We shall simply write $\int_B s\ d\mu$ for $\int_B s|B\, d\mu$ and call it *the integral of s over B*. Clearly,

since $A_k \cap B = \varnothing \Rightarrow \mu(A_k \cap B) = \mu(\varnothing) = 0$, (*) is also equal to $\sum_{k=1}^{n} \alpha_k \mu(A_k \cap B)$. So, we

have shown that **FACT 1.10** For non-negative measurable simple function $s : A \to \mathbb{R}$ with standard or acceptable representation $s = \sum_{k=1}^{n} \alpha_k \chi_{A_k}$ and $\varnothing \neq B \subseteq A$, $B \in$ A, we have

$$\int_B s\, d\mu \equiv \int_B s|B\, d\mu = \sum_{j=1}^{r} \alpha_j \mu(A_j \cap B)$$
$$= \int_A \left(\sum_{j=1}^{r} \alpha_j \chi_{A_j \cap B} \right) d\mu = \int_A \left(\sum_{j=1}^{r} \alpha_j \chi_{A_j} \chi_B \right) d\mu = \int_A \left((\sum_{k=1}^{n} \alpha_k \chi_{A_k}) \chi_B \right) d\mu$$
$$= \int_A s\chi_B \,.\ ///$$

REMARK 1.11 Compare FACT 1.10 with Definition 1.23, p. 20 of [8].

We assemble in the next theorem some well-known properties of the integral of non-negative measurable simple functions whose profs are mutatis mutandi as in the literature. And so we omit their proofs.

THEOREM 1.12 (i) Let $A, B, C \in A$ with $\varnothing \neq B \subseteq C \subseteq A$. For non-negative measurable simple $s : A \to \mathbb{R}$, $\int_B s\ d\mu \leq \int_C s\ d\mu$. [FACT 1.10, THEOREM 1.8 (iii) and the fact that $s\chi_B \leq s\chi_C$].

(ii) (See Proposition 2.3.2, p.62 of [1]) Let $s : A \to \mathbb{R}$ be a non-negative measurable simple function, and suppose $(s_n)_{n=1}^{\infty}$, $s_n : A \to \mathbb{R}$ $n = 1, 2, \ldots$, is an increasing sequence of non-negative measurable simple functions converging pointwise to s. Then,

$$\int_A s\, d\mu = \lim_{n\to\infty} \int_A s_n\ d\mu\,.$$

(iii) (See Proposition 2.1.7, p.54 of [1]) Consider the measurable space (X, A), and suppose $f : A \to \mathbb{R}^{e+}$ is a non-negative extended real-valued measurable function. Then, there exists an increasing sequence $(s_n)_{n=1}^{\infty}$ of non-negative measurable simple functions $s_n : A \to \mathbb{R}$ converging pointwise to f. ///

We furnish the proof of the next FACT 1.13. I have found the FACT useful but have not been able to locate its proof in the literature; it is only stated without proof in first two lines of the proof of Theorem VII.1.2, p.109 of [2].

FACT 1.13 (See first two lines of proof of Theorem VII.1.2, p.109 of [2]) Supp- ose $f, g : A \to \mathbb{R}^{e+}$ are non-negative extended real-valued measurable functions such that $f = g$ almost everywhere. If $s : A \to \mathbb{R}$ is a non-negative measurable simple func- tion satisfying $0 \le s \le f$, then there exists a non-negative measurable simple function $t : A \to \mathbb{R}$ satisfying

(i) $0 \le t \le g$, and

(ii) $s = t$ almost everywhere.

Proof Let $E = \{x \in A : f(x) \ne g(x)\}$. Clearly, $\{x \in A : f(x) \ne g(x)\} = \{x \in A : f(x) > g(x)\} \cup \{x \in A : g(x) > f(x)\}$ and so $E \in$ A by Proposition 2.1.2, p.49 of [1] being a union of two measurable sets. By hypothesis, therefore, $\mu(E) = 0$. If $E = \varnothing$, then $f = g$ everywhere, and clearly then, we have nothing to show as we simply take t as s. So, suppose $E \ne \varnothing$. Suppose also that s has the standard representation

$$s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \dots + \alpha_n\chi_{A_n}.$$

Suppose, possibly after suitable rearrangement,

$$A_1 \cap E \ne \varnothing, A_2 \cap E \ne \varnothing, \dots, A_q \cap E \ne \varnothing$$

but

$$A_{q+1} \cap E = \varnothing, A_{q+2} \cap E = \varnothing, \dots, A_n \cap E = \varnothing.$$

Since $\{A_1, A_2, \dots, A_n\}$ is a partition of A and $\varnothing \ne E \subseteq A$, it follows that $\{A_1 \cap E, A_2 \cap E, \dots, A_q \cap E\}$ is a partition of E. Then, we can write s as

$$s = \alpha_1\chi_{A_1-E} + \alpha_2\chi_{A_2-E} + \dots + \alpha_q\chi_{A_q-E} + \alpha_{q+1}\chi_{A_{q+1}} + \dots + \alpha_n\chi_{A_n} + \alpha_1\chi_{A_1\cap E} + \alpha_2\chi_{A_2\cap E} + \dots + \alpha_q\chi_{A_q\cap E}$$

is a, *non-necessarily* acceptable representation of s [E.g., if $A_1 \subseteq E$, say, (or more gene- rally, $A_1, A_2, \dots, A_r \subseteq E$) then $A_1 - E = \varnothing$; but an acceptable representing family is a partition of A and so members non-empty].

Now define $t : A \to \mathbb{R}$ by

$$t = \alpha_1\chi_{A_1-E} + \alpha_2\chi_{A_2-E} + \dots + \alpha_q\chi_{A_q-E} + \alpha_{q+1}\chi_{A_{q+1}} + \dots + \alpha_n\chi_{A_n} + 0\chi_{A_1\cap E} + 0\chi_{A_2\cap E} + \dots + 0\chi_{A_q\cap E}.$$

i.e. $t = \alpha_1\chi_{A_1-E} + \alpha_2\chi_{A_2-E} + \dots + \alpha_q\chi_{A_q-E} + \alpha_{q+1}\chi_{A_{q+1}} + \dots + \alpha_n\chi_{A_n}$.

Clearly, t is a non-negative measurable simple function [| Simple function $h : A \to \mathbb{R}$, $h = \sum_{k=1}^{m} \beta_k \chi_{B_k}$ is measurable $\Leftrightarrow B_1, B_2, \dots, B_m$ are measurable subsets of A. Note 1.2 (ii).|] satisfying $0 \le t \le g$ and $s = t$ almost everywhere. ///

NOTE 1.14 We shall soon run into an application of FACT 1.13 in the next section of this paper.

We move on to define *directly* the integral of **2** NON-NEGATIVE EXTENDED REAL-VALUED MEASURABLE $f : A \to \mathbb{R}^{e+}, A \in A$. Suppose $f : A \to \mathbb{R}^{e+}$ is an extended real-valued non-negative measurable function. Denote by $\text{MSF}^+(\le f)$ the set of all non-negative measurable simple functions $s : A \to \mathbb{R}$ satisfying $0 \le s \le f$. Clearly, $\text{MSF}^+(\le f) \ne \varnothing$ as the zero simple function $\kappa_0 : A \to \mathbb{R}$, $\kappa_0(x) = 0$ for all $x \in A$, belongs to $\text{MSF}^+(\le f)$. Besides, THEOREM

1.12(iii) guarantees the non-emptiness of $\text{MSF}^+(\le f)$. Taking cognizance of NOTE 1.7(i) and (ii), therefore **define** the *integral* of f, denoted $\int_A f\, d\mu$, by

$$\int_A f\, d\mu = \sup\{\int_A s\, d\mu : s \in \text{MSF}^+(\le f)\}.$$

Clearly, if $f : A \to \mathbb{R}$ is a non-negative measurable simple function there are now two definitions of the integral

$$\int_A f\, d\mu \qquad (*)$$

of f. Denote the meaning of (*) by the first definition by $\int_A^{\Delta} f\, d\mu$ and by the new definition by $\int_A^{\Delta\Delta} f\, d\mu$. Hence,

$$\int_A^{\Delta\Delta} f\, d\mu = \sup\{ \int_A s\, d\mu \ : s \in \text{MSF}^+(\leq f)\} \qquad (*)$$

But $f \in \text{MSF}^+(\leq f)$ and so $\int_A^{\Delta} f\, d\mu$ is inside the brackets of the R.H.S of (**), and there- fore, it follows from (**) that

$$\int_A^{\Delta} f\, d\mu \leq \int_A^{\Delta\Delta} f\, d\mu \qquad (1)$$

But by THEOREM 1.8(iii) (Monotonicity), since $s \in \text{MSF}^+(\leq f) \Rightarrow 0 \leq s \leq f$, then, $\int_A s\, d\mu \leq \int_A^{\Delta} f\, d\mu$ for all $s \in \text{MSF}^+(\leq f)$, and so $\sup_{s \in MSF^+(\leq f)} \int_A s\, d\mu \leq \int_A^{\Delta} f\, d\mu$.

That is,

$$\int_A^{\Delta\Delta} f\, d\mu \leq \int_A^{\Delta} f\, d\mu \qquad (2)$$

(1) and (2) give

$$\int_A^{\Delta} f\, d\mu = \int_A^{\Delta\Delta} f\, d\mu \,. ///$$

Consider extended real-valued non-negative measurable $f : A \rightarrow \mathbb{R}^{+e}$. Suppose $\varnothing \neq B \subseteq A$ is measurable, and consider the restriction

$f|B : B \rightarrow \mathbb{R}$, $(f|B)(b) = f(b)$ for all $b \in B$, of f to B.

The integral, $\int_B f\big|B\, d\mu$, of $f|B$ is called the *integral of f over B, and denoted* $\int_B f\, d\mu$.

Now consider non-negative measurable $f : A \rightarrow \mathbb{R}^{+e}$, with measurable $\varnothing \neq B \subsetneq A$. Consider the functions

$f\chi_B : A \rightarrow \mathbb{R}^{+e}$ and $f|B : A \rightarrow \mathbb{R}^{+e}$.

We show that

THEOREM 2.1 To every $s \in \text{MSF}^+(\leq f\chi_B)$ corresponds a $t \in \text{MSF}^+(\leq f|B)$, with same integral, and vice-versa

Proof Consider $s \in \text{MSF}^+(\leq f\chi_B)$ and so $s : A \rightarrow \mathbb{R}$ and $0 \leq s \leq f\chi_B$, from which follows that $s(x) = 0$ for $x \in A - B$ And that if s takes $\alpha \neq 0$, then $s^{-1}(\alpha) \subseteq B$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-3}$ be the distinct non-zero values of s. Hence, s has the representation

$$s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots + \alpha_{n-3}\chi_{A_{n-3}} + 0\chi_{A_{n-2}} + 0\chi_{A_{n-1}} + 0\chi_{A_n} \qquad (\nabla)$$

say, where

$A_1 \cup A_2 \cup \ldots \cup A_{n-3} \subseteq B$, $\alpha_1, \alpha_2, \ldots, \alpha_{n-3} \neq 0$, $A_k = s^{-1}(\alpha_k)$, $k = 1, 2, \ldots, n-3$, $A_{n-2} = A - B$,

$A_{n-1} = \{x \in B : f(x) = 0\} \subseteq B$. Clearly, because $s \in \text{MSF}^+(\leq f\chi_B)$, $s(x) = 0$ on A_{n-1}, $A_n = B - \bigcup_{k=1}^{n-2} A_k = \{x \in B : f(x) \neq 0, s(x) = 0\}$.

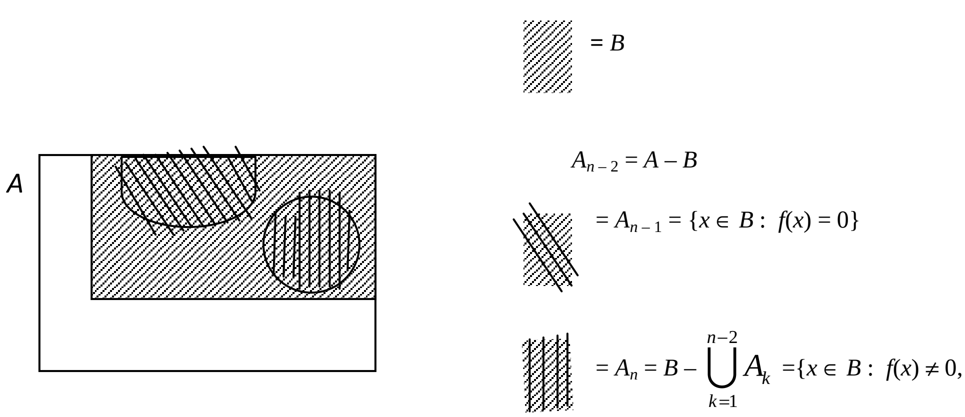

Fig.1. A Decomposition of A

The above Fig.1 shows *A*'s decomposition.
If $A_{n-1} = \emptyset = A_n$ then (∇) becomes the acceptable representation

$$s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots.. + \alpha_{n-3}\chi_{A_{n-3}} + 0\chi_{A_{n-2}} \qquad (\Sigma_1)$$

of *s* ; otherwise

with $A_{n-1} = \emptyset \neq A_n = B - \bigcup_{k=1}^{n-2} A_k$, (∇) now becomes the acceptable representation

$$s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots.. + \alpha_{n-3}\chi_{A_{n-3}} + 0\chi_{A_{n-2}} + 0\chi_{A_n} \qquad (\Sigma_2)$$

of *s*, while with $A_n = \emptyset \neq A_{n-1}$, ($\nabla$) now becomes the acceptable representation

$$s = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots.. + \alpha_{n-3}\chi_{A_{n-3}} + 0\chi_{A_{n-2}} + 0\chi_{A_{n-1}} \qquad (\Sigma_3)$$

of *s*.

Now, if (Σ_1) is true, define $t : B \to \mathbb{R}$ by

$$t = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots.. + \alpha_{n-3}\chi_{A_{n-3}} \qquad (\Pi_1)$$

and if (Σ_2) is true, define $t : B \to \mathbb{R}$ by

$$t = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots.. + \alpha_{n-3}\chi_{A_{n-3}} + 0\chi_{A_n} \qquad (\Pi_2)$$

and if (Σ_3) is true, define $t : B \to \mathbb{R}$ by

$$t = \alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots.. + \alpha_{n-3}\chi_{A_{n-3}} + 0\chi_{A_{n-1}} \qquad (\Pi_3)$$

Clearly, in all cases $t \in \mathrm{MSF}^+(\leq f\,|B)$ with (Π_1), (Π_2) and (Π_3), respectively, acceptable representations. Clearly, also, in all cases, by THEOREM 1.8(i),

$$\int_A s\, d\mu = \sum_{k=1}^{n-3} \alpha_k\, \mu(A_k) = \int_B t\, d\mu .$$

This concludes the proof in one direction. For the other direction suppose
$t : B \to \mathbb{R}$ and $t \in \mathrm{MSF}^+(\leq f\,|B)$
and suppose

$$t = \beta_1\chi_{B_1} + \beta_2\chi_{B_2} + \ldots.. + \beta_r\chi_{B_r} \qquad (\sigma_1)$$

is the standard representation of *t*. Then, define $s : B \to \mathbb{R}$ by

$$s = \beta_1\chi_{B_1} + \beta_2\chi_{B_2} + \ldots.. + \beta_r\chi_{B_r} + 0\chi_{A-B} \qquad (\sigma_2)$$

Clearly, (σ_2) is an acceptable representation of $s \in \mathrm{MSF}^+(\leq f\chi_B)$, and also

$$\int_B t\, d\mu = \sum_{j=1}^{r} \beta_j\, \mu(B_j) = \int_A s\, d\mu . ///$$

Employing the preceding theorem one deduces immediately from our *direct* definition of the integral of non-negative measurable $f : A \to \mathbb{R}^{+e}$ that

THEOREM 2.2 If $A, B \in \mathcal{A}$ and $\emptyset \neq B \subsetneq A$, then for extended real-valued non-negative measurable $f : A \to \mathbb{R}^{+e}$,

$$\int_A f\chi_B\, d\mu = \int_B f\big|B\, d\mu \equiv \int_B f\, d\mu . ///$$

NOTE 2.3 If $A = X$, THEOREM 2.2 is a definition of $\int_B f\, d\mu$ in the literature. And so if $A = X$, then we have shown that

THEOREM 2.4 (Our direct definition of $\int_B f\big|B$) $= \int_X f\chi_B\, d\mu$. ///

The promised application in NOTE 1.14 of FACT 1.13 is the proof of

THEOREM 2.5 (See Theorem VII.1.2(i), p.109of [2]) If $f, g : A \to \mathbb{R}^{+e}$ are extended real-valued non-negative measurable functions such that $f = g$ almost everywhere, then,

$$\int_A f\, d\mu = \int_A g\, d\mu .$$

Proof FACT 1.13 and THEOREM 1.8(ii).[Compare proof of (i) of Theorem 1.2., p.109 of [2]]. ///

The proofs of various theorems on the integral of non-negative extended real-valued measurable $f : A \to \mathbb{R}^{+e}$ (E.g., Monotonicity, Additivity, Positive Homogeneity, the Monotone Convergence Theorem, Fatou's Lemma, etc) are as in the literature mutatis mutandi. For an instance,

THEOREM 2.6 $f : X \to [0, \infty]$ measurable and $\bigcup_n B_n$ a countable disjoint union of non-empty measurable sets. Then, $\int_{\cup B_n} f \, d\mu = \sum_n \int_{B_n} f \, d\mu$.

Proof A careful adaptation of relevant parts of the proof of Proposition 1.7, p. 112 of [2], taking due cognizance of THEOREM 2.2. Or of the relevant parts of the proof of Corollary 2.4.2, p.71 of [1].///

3.0 Integral of Extended Real-Valued Measurable

$f : A \to \mathbb{R}^{+e}$ Again THROUGHOUT, $(X, \mathcal{A}, \mu)$ is a fixed measure space. This is done as in the literature : Write $f = f^+ - f^-$ and we say f *has an integral* if at least one of $\int_A f^+ \, d\mu$ and $\int_A f^- \, d\mu$ is finite and call $\int_A f^+ \, d\mu - \int_A f^- \, d\mu$ its *integral denoted* $\int_A f \, d\mu$. Call f *integrable* if both $\int_A f^+ \, d\mu$ and $\int_A f^- \, d\mu$ are finite and its *integral* denoted

$$\int_A f \, d\mu = \int_A f^+ \, d\mu - \int_A f^- \, d\mu$$

With $A = X$ we capture the definition in the literature. No theorem looses its claims and with proofs mutatis mutandi as in the literature.

4.0 Conclusion

Again, THROUGHOUT, $(X, \mathcal{A}, \mu)$ is a fixed measure space. We can also extend THEOREM 2.2 to

THEOREM 4.1 Let $A, B \in \mathcal{A}$ with $\varnothing \neq B \subsetneq A$. For extended real-valued integrable $f : A \to \mathbb{R}^{e}$.

$$\int_A f\chi_B \, d\mu = \int_B f\big|B \, d\mu \quad ///$$

With $A = X$ it becomes the definition of the literature's "the integral of f over B" (See p.65 of [1]). Finally, we show that

THEOREM 4.2 For integrable $f : A \to \mathbb{R}^{e}$, let

$$f^* : X \to \mathbb{R}^{e}, f^*(x) = \begin{cases} f(x), & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}.$$

Then, f^* is integrable, and our $\int_A f \, d\mu$ = lit's $\int_A f \, d\mu$ (Second paragraph, p.65of [1]) $\equiv \int_X f^* \, d\mu$.

Proof A number of comments are in order as we wade through the proof.

COMMENT 4.3 First, the measurability of f^* needs be established. We record this as a

LEMMA (Problem 3.21(b), p.69of [3]) 4.4 Suppose $E, D \in \mathcal{A}$, $\varnothing \neq E \subsetneq D$ and $f : E \to \mathbb{R}^{e}$ measurable. Define the function

$$g : D \to \mathbb{R}^{e}$$

$$x \mapsto \begin{cases} f(x), & x \in E \\ 0, & x \notin E \end{cases}$$

Then, f is measurable $\Leftrightarrow g$ is measurable. ///

COMMENT 4.5 Second paragraph of p.65 of [1] is not sure if the integral of f^* exists as signified by its phrase… (*if it exists*) …We show here that it actually exists.

Proof of Existence of $\int_X f^* \, d\mu$ **Case 1** f is a non-negative measurable simple function with standard representation $\alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \ldots + \alpha_n\chi_{A_n}$. If $A = X$ we have nothing to show. So suppose $A \neq X$. Then,

$$\text{our} \int_A f \, d\mu = \alpha_1\mu(A_1) + \alpha_2\mu(A_2) + \ldots + \alpha_n\mu(A_n) \qquad (\Sigma_1)$$

and,

$$\alpha_1\chi_{A_1} + \alpha_2\chi_{A_2} + \dots + \alpha_n\chi_{A_n} + 0\chi_{X-A}$$

is an acceptable representation of f^* and so

$$\int_X f^* \, d\mu = \alpha_1\mu(A_1) + \alpha_2\mu(A_2) + \dots + \alpha_n\mu(A_n) + 0\cdot\mu(X-A)$$
$$= \alpha_1\mu(A_1) + \alpha_2\mu(A_2) + \dots + \alpha_n\mu(A_n) = (\Sigma_1).$$

Case 2 f is a non-negative extended real-valued measurable function $f : A \to \mathbb{R}^e$. Again if $A = X$ we have nothing to show. So suppose $A \neq X$. Clearly,

$$\text{our} \int_A f \, d\mu = \sup\{ \int_A s \, d\mu \ : s \in \mathrm{MSF}^+(\leq f)\} \qquad (\Delta)$$

But any $s \in \mathrm{MSF}^+(\leq f)$ clearly has a *unique* extension $s^* : X \to \mathbb{R}$ to X, $s^* \in \mathrm{MSF}^+$ $(\leq f^*)$, and clearly also

$$\text{our} \int_A s \, d\mu = \int_X s^* \, d\mu \qquad (\Delta\Delta)$$

Similarly, clearly, any $s^* \in \mathrm{MSF}^+(\leq f^*)$ has a unique restriction to A, s. And clearly, $s \in \mathrm{MSF}^+(\leq f)$ and

$$\text{our} \int_A s \, d\mu = \int_X s^* \, d\mu \qquad (\Delta\Delta\Delta)$$

From (Δ), ($\Delta\Delta$) and ($\Delta\Delta\Delta$) now follows that our $\int_A f \, d\mu = \int_X f^* \, d\mu$.

Case 3 $f : A \to \mathbb{R}^e$ *integrable* or *integral exists* is now immediate from Case 2, employ- ing the decompositions $f = f^+ - f^-$ and $f^* = f^{*+} - f^{*-}$, and the easily checked fact that $f^{*+} = (f^+)^*$ and $f^{*-} = (f^-)^*$. And this completes the proof of THEOREM 4.2. ///

COMMENT 4.6 Let $\varnothing \neq A \subseteq X$, $A \in$ A, and $f : A \to \mathbb{R}^e$ an extended real-valued func- tion. Consider the trace $\mathrm{A}_A = \{A \cap E : E \in \mathrm{A}\}$ of A on A. Then, A_A is a σ-algebra on A and $\mu|\mathrm{A}_A$ is a measure on A_A. Denote $\mu|\mathrm{A}_A$ byμ_A. Then, $(A, \mathrm{A}_A, \mu_A,)$ is a measure space. We have.

THEOREM 4.7 With notation as in the preceding, f is A-measurable $\Leftrightarrow$ f is A_A-measurable.

Proof Clear!///
We also have

THEOREM 4.8 With notation as above, we can consider

(i) our $\int_A f \, d\mu$, and

(ii) (usual definition) $\int_A f \, d\mu$ w.r.t. the measure space $(A, \mathrm{A}_A, \mu_A,)$.

Then, our $\int_A f \, d\mu$ = (usual definition) $\int_A f \, d\mu$ w.r.t. the measure space $(A, \mathrm{A}_A, \mu_A,)$. ($\Delta$)

Proof Check the equality in (Δ) first for f a non-negative simple measurable function. Then, with this confirmed, check the equality for f a non-negative extended real-valued measurable function. Finally check the equality for f an arbitrary extended real-valued measurable function. /// [| Compare 4.7.6, p.187/188 of [4] |]. So, we now have with THEOREM 4.2.

THEOREM 4.9 With notation as above

lit's $\int_A f \, d\mu$ = our $\int_A f \, d\mu$ = (usual definition) $\int_A f \, d\mu$ w.r.t. the measure space $(A, \mathrm{A}_A, \mu_A,)$. ///

REMARK 4.10 Compare the second equality in the preceding THEOREM 4.9 with [1, lines 6 – 10 of the proof of theorem 5.1.13, p. 207]. From THEOREM 4.9 follows that **COROLLARY 4.11** With notation as above

lit's $\int_A f \, d\mu$ = (usual definition) $\int_A f \, d\mu$ w.r.t. the measure space (A, A_A, μ_A). ///

Example 4.12 Exercise 1, p.188 of [4] is a restatement of our COROLLARY 4.11 above for Lebesgue integrable $f : [a, b] \to \mathbb{R}$.

COMMENT 4.13 The last sentence of first paragraph of page 134 of [1] in its proof of the Radon-Nikodym Theorem reads: *The function* $g : X \to [0, \infty)$ *that agrees on each* B_n *with* $g_n : B_n \to [0, \infty)$ *is then the required function*. We furnish a proof. We refer to the proof of Theorem 4.2.2, p. 132 –134 of [2]. As claimed by the author: *For each n the first part of this proof provides an A-measurable function* $g_n : B_n \to [0, \infty)$ *such that* $\nu(A) = \int_A g_n \, d\mu$ *holds for each A-measurable subset A of* B_n

[|Note: Of course if $A = \emptyset$, $\int_A g_n \, d\mu$ does not make sense, but then $\nu(A) = 0$|]. And, therefore, if $\emptyset \neq A \in \mathcal{A}_{B_n}$ ($\equiv$ the trace $\mathcal{A}|B_n$ of $\mathcal{A}$ on B_n) there exists $g_n^{\Delta} : (B_n, \mathcal{A}_{B_n}, \mu) \to \mathbb{R}^+$ such that

$$\nu(A) = \int_{A\cap B_n} g_n^{\Delta} \, d\mu = \int_{B_n} g_n^{\Delta} \chi_{A\cap B_n} \, d\mu \qquad (\rho_1)$$

Now, let $\emptyset \neq A \in \mathcal{A}$. Suppose $\emptyset \neq A\cap B_n \in \mathcal{A}_{B_n}$. By ($\rho_1$) therefore, there exists $g_n^{\Delta} : (B_n, \mathcal{A}_{B_n}, \mu) \to \mathbb{R}^+$ such that

$$\nu(A\cap B_n) = \int_{A\cap B_n} g_n^{\Delta} \, d\mu = \int_{B_n} g_n^{\Delta} \chi_{A\cap B_n} \, d\mu \qquad (\rho_2)$$

Now, let $g_n^{\Delta} \chi_{A\cap B_n} = h_n$. And so, from ($\rho_2$),

$$\nu(A\cap B_n) = \int_{B_n} h_n \, d\mu \qquad (\rho_3)$$

Hence,

$$\nu(A) = \nu\left[\left(\bigcup_n B_n\right) \cap A\right]$$

$= \Sigma_n \, \nu(A\cap B_n)$, which by ($\rho_3$),

$$= \sum_n \int_{B_n} h_n \, d\mu ,$$

which by our THEOREM 4.2,

$$= \sum_n \int_X h_n^* \, d\mu$$

which by the Monotime Convergence Theorem(Applicable if $A\cap B_n \neq \emptyset$ for infinitely many n)

$$= \int_X \left(\sum_n h_n^*\right) d\mu$$

$g = \sum_n h_n^*$ is the function being claimed. ///

REMARK 4.14 The discussion in COMMENT 4.13 taking cognizance of THEOREM 4.2, THEOREM 4.8 and COROLLARY 4.11 strongly justifies our definition of the integral of measurable $f : A \to [-\infty, \infty]$ *directly*, for $\emptyset \neq A \subseteq X$, $A \in \mathcal{A}$. It is hoped that new books on the integral will take to the presentation of this paper.

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