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Response Analysis of a Beam Subjected To a Moving Load Using Green's Function Solution.

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Abstract

Dynamical response of beams under moving loads were investigated using a Green function solution.

The partial differential equation governing the model is reduced to an Ordinary Differential Equation. This was solved using Mathematical Software (maple).

The results were presented in a graphical and tabular form.

It was observed that the response amplitude of the beam decreases as the mass of the load increases.

Also it was observed that the amplitude of the beam increases as the value of the length increases.

Keywords: Beams, moving loads, Dynamic response, Amplitude, Green function.

1.0 Introduction

The structures in which the dynamic effects of moving load play the most important role are bridges, particularly the railway bridges whose behaviour under these effects have been studied and will continue to be studied because of the significance of the safety of road and rail transportation [1-15].

Also the concept of beam and slabs on elastic foundation has been extensively used by geotechnical, pavement and railroad engineers for foundation design and analysis. The analysis of structures resting on elastic foundation is usually based on a relatively simple model of the foundation response in applied loads.

A beam is the simplest kind of bridge. In the olden days, they took the form of log across a stream, but today they are more seen as large box steel girder bridges. A beam is simply refers to anything that provide a passage over some sort of obstacle such as river, road, a dam, set of rail track and so on.

A modern bridge is likely to span a distance up to 60m, while a modern arch bridge can safely span up to 240m. A suspension bridge is referred to as the pinnacle of technology and capable of spanning up to 2,100m.

What makes an arch bridges to span greater distance than a beam bridge or a suspension bridge to span a distance seven times that of an arch bridge?. This depends on how each bridge type deals with the forces called compression and tension.

Transport Engineering structures are subjected to loads that vary in time and space; such loads are moving loads, moving trains, trucks, cars or cranes. In the case of constant stationary loads, the reactions, stress, and deformation at a particular point are constant. If the loads are moving then the load effects become a variable function of the position of the load.

Moving loads can either be concentrated or distributed. This research work deals with the problems involving concentrated Load.

The structures on which these moving loads move are usually modeled by elastic beams, plates or shells.

Elastic beams can be defined as structural members that react to forces applied to it transversely or laterally to their axes.

The boundary conditions most frequently encountered in analyzing vibration of beams are fixed end, simply supported end and free end conditions. These are called classical or ideal end conditions.

Generally, the analysis of bending of beams on an elastic foundation is developed on the assumption that the forces of the foundation are proportional at every point to the deflection of the beam at the point. The vertical deformation

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characteristics of the foundation are defined by means of these springs is known as the modules of subgrade reaction (K_o). The simple representation of elastic foundation was introduced by Winkler in 1867. The approach introduces a linear algebraic relationship between the normal displacement of the structure and the contact pressure. The Winkler model represents the soil medium by a set of mutually independent spring element. Such approach grants simplicity in obtaining closed form solution .Moreover, it gives the chance of obtaining a non-linear behaviour with lower computational efforts compared to other methods.

The Winkler model which has been originally developed for the analysis of rail road tracks is very simple but does not accurately represents the characteristics of many practical foundations. One of the most important deficiencies of the Winkler model is that a displacement discontinuity appears between the loaded and the unloaded part of the foundation surface. In reality, the soil surface does not show any discontinuity.

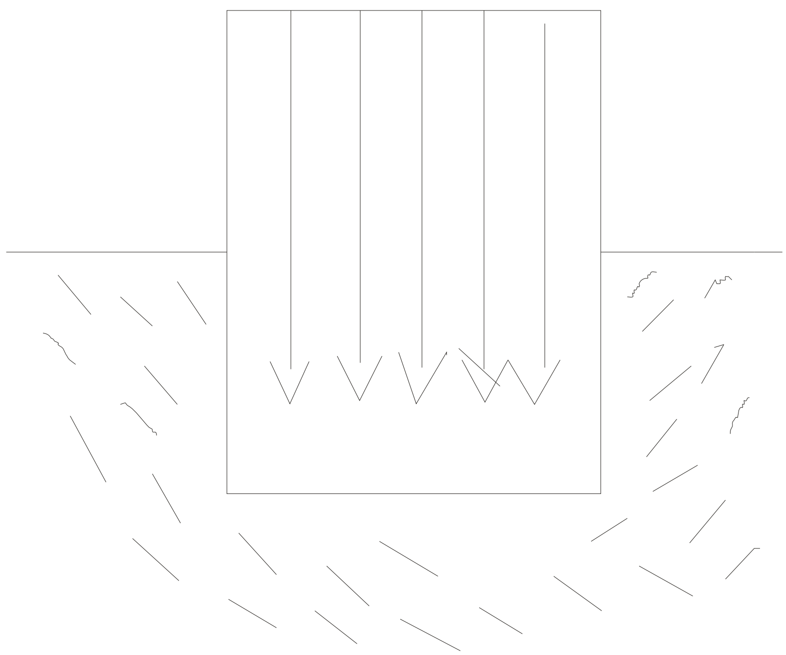

Figure1 :WINKLER FOUNDATION.

The traditionally way to overcome the deficiency of Winkler model is by introducing some kind of interaction between the independent springs by visualizing various types of interconnections such as flexural elements (beam in 1 dimension), plates in 2- dimension etc. It is well known that when loads move on structural membranes the resistance of the bending produces two effects which cause the structure to vibrate continuously. These two effects in the field of structural dynamics are termed the moving force effect and the moving mass effects. In all of the aforementioned studies, the damping term in the governing differential equation of the motion is neglected and the effect of elastic foundation of the non- uniform stiffness was not investigated. The study therefore investigates the response of beams under a moving load using Green function solution.

Purpose of the Study

(1). To find the effect of mass on concentrated moving mass and concentrated moving force.
(2). To find the effect of length on concentrated moving loads.
What is a beam? A beam is a horizontal structural element that is capable of withstanding load primarily by resisting bending moment (A bending moment exists in a structural element when a moment is applied to the element so that the element bends).
Beams generally carry vertical gravitational forces but can also be used to carry horizontal loads (i.e. loads due to an earthquake or wind). The loads carry by a beam are transferred to columns, walls or girders, which then transfer the force to adjacent structural compression members. Beams are characterized by their profile (the shape of their cross-section) their length and their material. This is commonly used in steel-frame buildings of bridges. For many years the dynamics design of railway bridges was dominated by the problem of “hammer blow” which consisted of pulsating forces generated by the balance weights on the driving wheels of steam locomotives.

TYPES OF BEAM
(1). A simply supported beam (simple beam)
(2). Cantilever beam (fixed end beam)
(3). Beam with an overhang
(4). Suspension beams
What is load? What we are referring to structural loads and it can actually be defined as forces, deformations or accelerations applied to a structure or its components.
Loads cause stresses, deformations and displacement in structures. Assessment of their effects is carried out by the methods of structural analysis.
TYPES OF LOADS
(1). Concentrated load (single force)
(2).Distributed load: loads measured by their intensity
(a).Uniformly distributed load
(b). Linearly varying distributed load
(c). Partially distributed load
(3). Couple.

Mathematical Formulation Of The Problem

The governing equation of a flexible beam subject to a concentrated force (See Figure 2) can be given by:

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\mu\frac{\partial^2 y(x,t)}{\partial t^2}=F(x,t) \qquad (1)$$

Where:
$y(x,t)$ represents the deflection of the beam
x represents the traveling direction of the moving load
t represents time.
Also, EI is the rigidity of the beam
E is Young's modulus of elasticity
I is the cross sectional moment of inertial of the beam
μ is the mass per unit length of the beam.
The beam length is L, traveling load velocity is v. the boundary conditions and the initial conditions for the general beam (Figure 1) are:

$$\left.\begin{aligned}&\frac{\partial^3 y(x,t)}{\partial x^3}=kly(x,t)\\&\frac{\partial^2 y(x,t)}{\partial x^2}=kt\frac{\partial y(x,t)}{\partial x},\ for\ \ x=0\ \ and\ \ x=l.\\&y(x,t)=\frac{\partial y(x,t)}{\partial t}=0\end{aligned}\right\} \qquad (2)$$

Where kl and kt are linear and twisting spring constants, preventing vertical motion and, in the $x-y$ plane, rotation of the beam ends, respectively.
$F(x,t)$ is the external load and, for a moving concentrated load case, can be given by:

$$F(x,t)=Mg\left[1-\frac{1}{g}\frac{d^2 y(x,t)}{dt^2})\delta(x-vt)\right] \qquad (3)$$

Where
M is the mass of the load, g is acceleration due to gravity
$\delta(x-vt)$ is the Dirac-delta function define to be zero everywhere except $x=vt$. I.e $\delta(x,vt)=0,\quad x\neq vt$
The moving load is assumed to move with constant velocity. Consequently, the convective acceleration operator is defined as

$$\frac{d^2}{dt^2}=\frac{\partial^2}{\partial t^2}+2v\frac{\partial^2}{\partial x\partial t}+v^2\frac{\partial^2}{\partial x^2} \qquad (4)$$

Using the dynamic Green function, the solution of equation (1) can be written as:

$$y(x,t) = G(x,u)p \tag{5}$$

Where $G(x,u)$ is the solution of the differential equation.

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \mu\frac{\partial^2 y(x,t)}{\partial t^2} = Mg\left[1 - \frac{1}{g}\frac{d^2 y(x,t)}{dt^2}\right]\delta(x - vt) \tag{6}$$

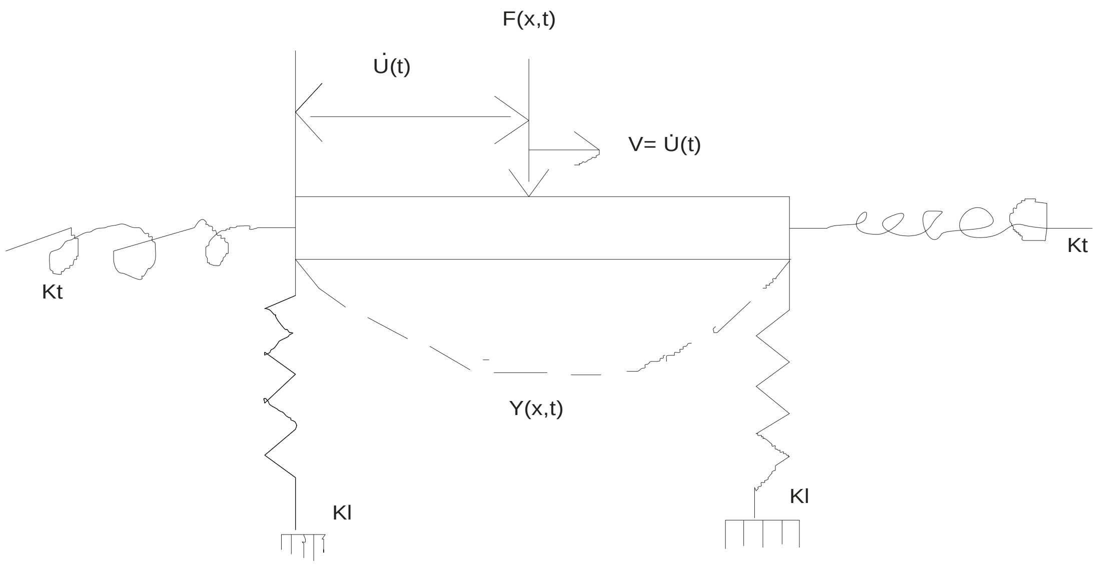

Figure 2 :Moving mass on a beam with general boundary condition.

METHOD OF SOLUTION

Evidently, a closed form solution of the partial differential equation (6) does not exist.
A Green function solution of ordinary differential equation of the governing equation in terms of the normal mode is in form:

$$\left.\begin{aligned}
&y(x,t) = G(x,u)p \\
&y'(x,t) = X_n'(x)U_n(u)P = \frac{\partial y(x,t)}{\partial x} \\
&y''(x,t) = X_n''(x)U_n(u)P = \frac{\partial^2 y(x,t)}{\partial x^2} \\
&y'''(x,t) = X_n'''(x)U_n(u)P = \frac{\partial^3 y(x,t)}{\partial x^3} \\
&y''''(x,t) = X_n''''(x)U_n(u)P = \frac{\partial^4 y(x,t)}{\partial x^4} \\
&\dot{y}(x,t) = X_n(x)\dot{U}_n(u)P = \frac{\partial y(x,t)}{\partial t} \\
&\ddot{y}(x,t) = X_n(x)\ddot{U}_n(u)P = \frac{\partial^2 y(x,t)}{\partial t^2}
\end{aligned}\right\} \tag{7}$$

Substituting equation (3) and equation (4) into equation (1) and assuming the flexural rigidity EI, and the mass per unit length μ do not vary with the position X along the span L.

Equation (1) becomes

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\mu\frac{\partial^2 y(x,t)}{\partial t^2}=Mg\left[1-\frac{1}{g}\left(\frac{\partial^2 y(x,t)}{\partial t^2}+2V\frac{\partial^2 y(x,t)}{\partial x\partial t}+V^2\frac{\partial^2 y(x,t)}{\partial x^2}\right)\right]\delta(x-vt) \tag{8}$$

Therefore, substituting equation (7) into equation (8), we have

$$EIX_n^{IIII}(x)U_n(u)P+\mu X_n(x)\ddot{U}_n(u)P=Mg\left[1-\frac{1}{g}\left(\frac{\partial^2 y(x,t)}{\partial t^2}+2V\frac{\partial^2 y(x,t)}{\partial x\partial t}+V^2\frac{\partial^2 y(x,t)}{\partial x^2}\right)\right]\delta(x-vt) \tag{9}$$

Let

$$EIX_n^{IIII}(x)=\mu w_n^2 X_n(x)$$

$$\mu W_n^2 X_n(x)U_n(u)+\mu X_n(x)\ddot{U}_n(u)=Mg\left[1-\frac{1}{g}\left(\frac{\partial^2 y(x,t)}{\partial t^2}+2V\frac{\partial^2 y(x,t)}{\partial x\partial t}+V^2\frac{\partial^2 y(x,t)}{\partial x^2}\right)\right]\delta(x-vt) \tag{10}$$

Multiplying both sides of equation (10) by X_k (x) and integrating along the entire length of the beam

$$\int_0^L \mu W_n^2 X_n(x)U_n(u)X_k(x)dx+\int_0^L \mu X_n(x)\ddot{U}_n(u)X_k(x)dx=\int_0^L Mg\left[1-\frac{1}{g}\left(\frac{\partial^2 y(x,t)}{\partial t^2}+2V\frac{\partial^2 y(x,t)}{\partial x\partial t}+V^2\frac{\partial^2 y(x,t)}{\partial x^2}\right)\right]\delta(x-vt) \tag{11}$$

Using orthogonality conditions. That is,

$$\int_0^L X_n(x)X_K(x)dx=\begin{pmatrix}0, n\neq k\\ \alpha, n=k\end{pmatrix} \tag{12}$$

$$\alpha\mu W_n^2 U_n(u)+\alpha\mu\ddot{U}_n(u)=\int_0^L Mg\left[1-\frac{1}{g}\left(\frac{\partial^2 y(x,t)}{\partial t^2}+2V\frac{\partial^2 y(x,t)}{\partial x\partial t}+V^2\frac{\partial^2 y(x,t)}{\partial x^2}\right)\right]\delta(x-vt)dx \tag{13}$$

$$W_n^2 U_n(u)+\ddot{U}_n(u)=\frac{1}{\alpha\mu}\int_0^L Mg\left[1-\frac{1}{g}\left(\frac{\partial^2 y(x,t)}{\partial t^2}+2V\frac{\partial^2 y(x,t)}{\partial x\partial t}+V^2\frac{\partial^2 y(x,t)}{\partial x^2}\right)\right]\delta(x-vt)dx \tag{14}$$

$$W_n^2 U_n(u)+\ddot{U}_n(u)=\frac{1}{\alpha\mu}\left[\begin{array}{l}\int_0^L MgX_k(x)\delta(x-vt)dx-\int_0^L MX_n(x)\ddot{U}_n(u)X_k(x)\delta(x-vt)dx-\\ \int_0^L 2MVX_n^I(x)\dot{U}_n(u)X_k(x)\delta(x-vt)dx-\int_0^L MV^2X_n^{II}(x)U_n(u)\delta(x-vt)X_k(x)dx\end{array}\right] \tag{15}$$

$$\ddot{U}_n(u)+W_n^2U_n(u)=$$
$$\frac{1}{\alpha\mu}\left[\begin{array}{l}\int_0^L MgX_k(x)\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\cos\frac{n\pi x}{L}\right)dx-\int_0^L MX_n(x)\ddot{U}_n(u)X_k(x)\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\cos\frac{n\pi x}{L}\right)dx-\\ \int_0^L 2MVX_n^I(x)\dot{U}_n(u)X_k(x)\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\cos\frac{n\pi x}{L}\right)dx-\int_0^L MV^2X_n^{II}(x)U_n(u)X_k(x)\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\cos\frac{n\pi x}{L}\right)dx\end{array}\right] \tag{16}$$

$$\ddot{U}_n(u)+W_n^2U_n(u)=\frac{1}{\alpha\mu}\left(P_1-P_2\ddot{U}_n(u)-P_3\dot{U}_n(u)-P_4U_n\right) \tag{17}$$

Where

$$\left.\begin{aligned}
P_1 &= \int_0^L MgX_k(x)\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\cos\frac{n\pi x}{L}\right)dx \\
P_2 &= \int_0^L MX_n(x)\ddot{U}_n(u)X_k(x)\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\cos\frac{n\pi x}{L}\right)dx \\
P_3 &= \int_0^L 2MVX_n^{\prime}(x)\dot{U}_n(u)X_K(x)\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\cos\frac{n\pi x}{L}\right)dx \\
P_4 &= \int_0^L MV^2X_n^{\prime\prime}(x)U_n(u)X_K(x)\left(\frac{1}{L}+\frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi ut}{L}\cos\frac{n\pi x}{L}\right)dx
\end{aligned}\right\} \qquad (18)$$

NUMERICAL METHODS AND DISCUSSION OF RESULT

To illustrate the foregoing analysis, uniform beams of length 15m, 20m and 25m and mass 3kg, 6kg and 9kg respectively were considered.

$$\mu = 75, EI = 2785Nm^{-1}, K = 1,2,3...V = 3.3ms^{-1}, G = 10ms^{-2}, \pi = 3.142, n = 1,2,3...$$

The results were shown in tabular form and plotted curves below for the various values of masses and lengths.

TABLE 1: Various values of Masses and lengths

s/n	t(sec)	Y(x,t) at m=3kg	Y(x,t) at m=6kg	Y(x,t) at m=9kg
1	0	0.0000	0.0000	0.0000
2	0.1	-0.4185	-0.6813	-0.9479
3	0.2	-1.2253	-1.8291	-2.4456
4	0.3	-2.4098	-3.4241	-4.4616
5	0.4	-3.9586	-5.4413	-6.9564
6	0.5	-5.8562	-7.8505	-9.8836
7	0.6	-8.0854	-10.6180	-13.1922
8	0.7	-10.6281	-13.7075	-16.8287
9	0.8	-13.4665	-17.0816	-20.7394
10	0.9	-16.5829	-20.7035	-24.8721
11	1	-19.9613	-24.5383	-29.1789
12	1.1	-23.5872	-28.5548	-33.6171
13	1.2	-27.4488	-32.7264	-38.1515
14	1.3	-31.5370	-37.0325	-42.7553
15	1.4	-35.8458	-41.4595	-47.4107
16	1.5	-40.3724	-46.0012	-52.1099
17	1.6	-45.1173	-50.6595	-56.8547
18	1.7	-50.0840	-55.4442	-61.6562
19	1.8	-55.2790	-60.3731	-66.5342
20	1.9	-60.7113	-65.4710	-71.5162
21	2	-66.3919	-70.7692	-76.6357

Table 2: Various values of Masses and lengths

s/n	t(sec)	Y(x,t) at L=15m	Y(x,t) at L=20m	Y(x,t) at L=25m
1	0	0.0000	0.0000	0.0000
2	0.1	-0.4185	-0.4250	-0.3597
3	0.2	-1.2253	-1.1296	-0.8852
4	0.3	-2.4098	-2.1029	-1.5666
5	0.4	-3.9586	-3.3325	-2.3929
6	0.5	-5.8562	-4.8049	-3.3528
7	0.6	-8.0854	-6.5060	-4.4351
8	0.7	-10.6281	-8.4217	-5.6289
9	0.8	-13.4665	-10.5385	-6.9243
10	0.9	-16.5829	-12.8441	-8.3125
11	1	-19.9613	-15.3276	-9.7866
12	1.1	-23.5872	-17.9805	-11.3417
13	1.2	-27.4488	-20.7964	-12.9748
14	1.3	-31.5370	-23.7720	-14.6854
15	1.4	-35.8458	-26.9067	-16.4754
16	1.5	-40.3724	-30.2029	-18.3490
17	1.6	-45.1173	-33.6660	-20.3125
18	1.7	-50.0840	-37.3042	-22.3742
19	1.8	-55.2790	-41.1279	-24.5440
20	1.9	-60.7113	-45.1499	-26.8332
21	2	-66.3919	-49.3845	-29.2538

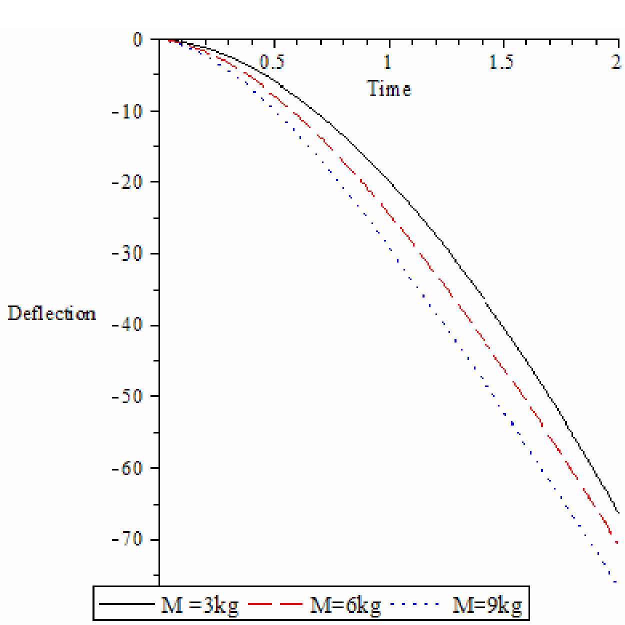

Fig 3: Graph of reflection against Time for different values of masses M

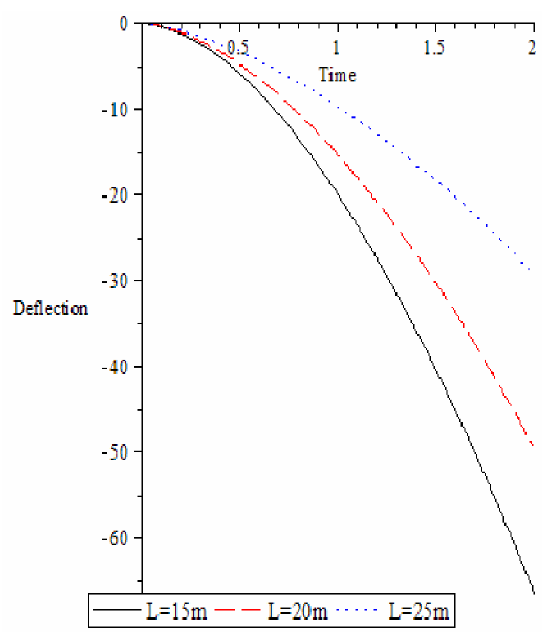

Fig 4: Graph of reflection against Time for different values of Length L

Conclusion

In conclusion, dynamical response of Beams under moving load using Green function solution is considered.

The beam is assumed to be simply supported at both ends.

In this problem, using a Green function solution for the dynamic deflection in terms of normal modes, the equation governing the model is reduced to a set of Ordinary differential equation.

Figure 3 shows that the response amplitude of the beam decreases as the value of the mass increases, while Figure 4 shows that the response amplitude of the beam increases as the value of the length increases.

References

[1] E. Savin.(2001). Dynamics amplification factor and response spectrum for the evaluation of vibrations of beam under successive moving loads. Journal of sound and vibrations 248 (2); 267-288.
[2] L. Frybal (1972): Vibration of solids and structures under moving loads. Prague Research Institute of Transport.
[3] G. Muscolino and A. Palmeri.(2007). Response of beams resting on visco elastically damped foundation to a moving oscillation. International journal of solids and structures, 44(5): 1317 - 1336.
[4] Gbadeyan, J.A and M.S Dada, 2006. Dynamic response of a mindlun elastic rectangular plate under a distributed moving mass. Int. J.mech. Sci, 48:323-340.
[5] Barbara Lazzaru, Roberta Nibbi (2011): On the exponential decay of the Euler-Bernoulli beam with boundary energy dissipation.
[6] A. M Krall, (1989): Asymptotic stability of the Euler-Bernoulli beam with boundary control. Journal of Mathematical Analysis and Applications 137, 288-295.
[7] G. G. Adams.(1995). Critical speeds and the response of a tensioned beam on elastic Foundation to repetitive moving loads. Int. jour. Mech. Sci., 7; 773-781.
[8] H. Saito, S. Chonan and O, Kawanobe.(1980). Response of an elastically supported plate strip to a moving load. Journal of sound and vibration, 71(2); 191 - 199.
[9] J.A. Gbadeyan and S. T. Oni (2007). the effects of linearly varying distributed moving loads on beams. Journal of Engineering and applied sciences, 2(6); 1006 - 1011.
[10] Lin Y.H: Vibration analysis of the beams traversed by uniform partially distributed moving masses. Journal of sound [and vibration, VoIl.I19.
[11] Mahmoud, M.A Abouzaid, 2002. Dynamic response of a beam with a crack subjected to a moving mass. J sound vibration, 250:291-603.
[12] (Michactsos, G.T ans A.N.Kounads, 2011. The Effect of centripetal and coriolis forces on the dynamic response of light bridges under moving loads. J. vibration control 7:315-326.
[13] Michactsos, G.T,2002. Dynamic behavior of a single-span beam subjected to loads moving with variable speeds. J. Sound Vibration, 258:359-372.
[14] Pilkey, W.D and O.H Pilkey, 1974. Mechanics of solids, Quantum publishers, INC. New York.
[15] S.Sadiku and H.H.E Leipholz (1987): on the dynamics of elastic system with moving concentrated mass. Ingenieur Archive, 57:223-242.