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Mathematical Modelling of Electrical Networks

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Abstract

Application of linear algebra to other branches of science, engineering and economics or elsewhere occurs via the need to solve such system of linear equation. The main aim of linear algebra is to find the most economical way of manipulating and solving such systems. There have been studies on application of linear algebra in electrical networks. In this paper, we shall show how a matrix theory played an important role in characterizing connections both in space and in electric circuits. We also make an extension on when given a nodal incidence matrix, one should by analysis draw a corresponding electrical networks. We model an electrical circuit problem that yields a unique solution. This is achieved with the help of the Kirchhoff`s current law (KCL) and Kirchhoff`s voltage law (KVL).

Keywords: Reference node, loop, network, nodal incidence matrix, mesh incidence matrix, grounded node.

1.0 Introduction

The introduction and the development of the notion of a matrix and the subject of linear algebra followed the development of determinants, which arose from the study of coefficients of system of linear equations.

Linear algebra is a branch of mathematics concerned with the study of the theory and application of linear systems of equations (briefly called linear systems), linear transformations and eigen-value problems as they arise from electrical networks, frameworks in mechanics, curve fitting and other optimization problems, processes in statistics and systems of differential equations are examples of applications of linear algebra. Among other applications of linear algebra, this paper deals mainly with application of linear algebra in modelling electrical networks. Linear algebra entered applied mathematics more than sixty years ago and is of increasing importance in various fields, especially in science, engineering and social sciences [1]. Kirchhoff's current law was used in [2, 4, 5] for the analysis of electrical circuits. Basic definitions of some terms related to matrix and linear algebra were also obtained in [3]. Gauss-Jordan and Gaussian elimination is used to solve the systems of the linear equations generated [6].

Sylvester in 1948 introduced the term "matrix" which was the Latin word for the womb as a name for an array of numbers [7].A Matrix has two important properties. It acts as compact store information and it allows assemblies of information to be handled simultaneously instead of one item at a time. Fredrich Gauss in 1800 develop Gaussian elimination method and use it to solve least squares problems in celestial bodies (stars, moon, planet etc) [7]. Matrix algebra was nurtured by the work of Arthur Cayley in1885. Cayley studied compositions of linear transformation ST as the product of the matrix for S times the matrix T. He went on to study the algebra of these compositions including matrix inverses.

Alan Turing introduced the LU decomposition of a matrix in 1948. The L is a lower triangular matrix with 1's on the diagonal and the U is an echelon matrix. It is common to use LU decomposition in a solution of a sequence of system of linear equations each having the coefficient matrix [6, 7].

1. Preliminaries

Linear equations: An expression of the form

$a_1\ x_1 + a_2\ x_2 + \ldots + a_n\ x_n$ ……………………………………..……….(1)

is called a linear equation Where $a_{i,}$b ϵ ℝ and the $x_{i,}$ iϵℕare unknowns. The scalars a_i are called the coefficients of the x_i respectively and b is called the constant term or simply constant of the equation.

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A set of values for the unknowns say $x_1 = k_1$, $x_2 = k_2$,…, $x_n = k_n$ is a solution of (1). If the statement obtained by substituting k_i for x_i $a_1\ k_1 + a_2\ k_2 + \ldots + a_n\ k_n$ = b is true. This set of values is then said to satisfy the equation. If there is no ambiguity about the position of the unknowns in the equation, then we denote this solution by simply the n-tuple U= $(k_1, k_2, \ldots, k_n)$

Linear systems of equations: A linear systems of m equations in n unknowns $x_1, x_2, \ldots, x_n$ is a set of equations of the form

s

$$\left.\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The a_{jk} are given numbers, which are called the coefficients of the system. The $b_i's$ are also given numbers. If the $b_i's$ are all zero, then (2) is called a homogeneous system. If at least one b_i is not zero, then (2) is called non homogeneous system. A solution of (2) is a set of numbers $x_1, \ldots, x_n$ that satisfy all the m equations. If the system (2) is homogeneous, it has at least the trivial solution $x_1 = 0, \ldots, x_n$=0.

The system of linear equations (2) is said to be over determined if the number of unknowns is less than the number of equations.

The system of linear equations (2) is called under determined if the number of unknowns is more than the number of equations.

The system of equations (2) is said to be determined if the number of unknowns is equal to the number of equations, that is m= n. The linear system of equations (2) is said to be consistent if it has a unique solution, otherwise, the linear system is said to be inconsistent (that is has infinitely many solutions) .

Coefficient Matrix, Augmented Matrix: From the definition of matrix multiplication, we see that the m equations of (2) can be written as single vector equation AX = b, where the coefficient matrix A= $[a_{jk}]$ is the m×n matrix

$$A = \begin{bmatrix} a_{11}a_{12} \ldots., a_{1n} \\ a_{21}\ a_{22} \ldots., a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m2} \ldots\ldots\ldots\ldots a_{mn} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_m \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} a_{11}a_{12} \ldots., a_{1n} & b_1 \\ a_{21}\ a_{22} \ldots., a_{2n} & b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. & \\ & \\ a_{m2} \ldots\ldots\ldots\ldots a_{mn} & b_m \end{bmatrix}$$

is called the augmented matrix of the system (2). D is obtained by augmenting A by the column b. The matrix D determines the system (2) completely because it contains all the given numbers appearing in (2).

Kirchhoff's First Law: the sum of the currents flowing in to a node is equal to the sum of the current flowing out.

Kirchhof's Second Law: The algebraic sum of the voltage drops around a close loop is equal to the total voltage in the loop.

Electrical networks is the arrangement of a system of interconnected electrical conductors such as resistors, inductors, capacitors, transmission lines and voltage sources that cover a desired region.

1. Application of linear algebra in modelling electrical networks

Nodal Incidence Matrix

The Figure 3.1 shows an electrical network in an open space having six branches and four nodes (junction). Where A, B and C represent the nodes and each node generates a row of the nodal incidence matrix. The numbers 1to 6 are the branches.

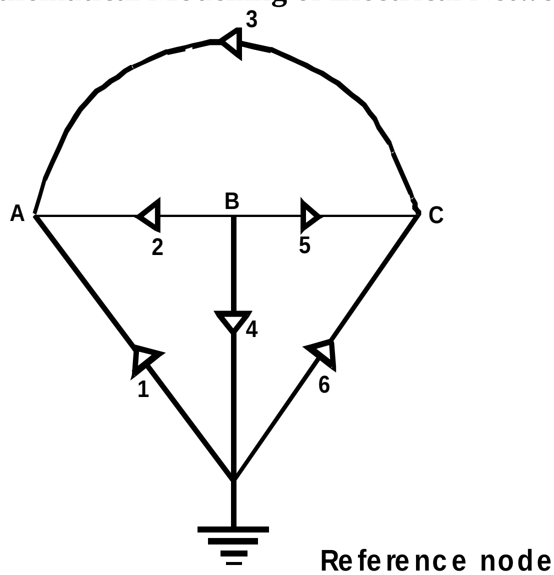

Fig 3.1 **An electrical network in an open space having six branches and three nodes**

To generate the entries of the matrix, we define

$$a_{jk} = \begin{cases} +1, if\ branch\ k\ leaves\ node\ (j) \\ -1, if\ branch\ k\ enters\ node\ (j) \\ 0, if\ branch\ k\ does\ not\ touch\ node\ (j) \end{cases}$$

Then, $A = [a_{jk}]$ is called the nodal incidence matrix of the electrical network in Figure 3.1

Branches 1 2 3 4 5 6

$$\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

Where the first, second and third row is obtained at the node A, B, and C respectively.

Mesh Incidence Matrix

A mesh is a loop without a branch in its interior or exterior. A network can also be characterized by the mesh incidence matrix. In the Fig. 3.2, the numbers in the squares are the loops and each loop generates a row of the mesh incidence matrix while the unsquared numbers represent the branches.

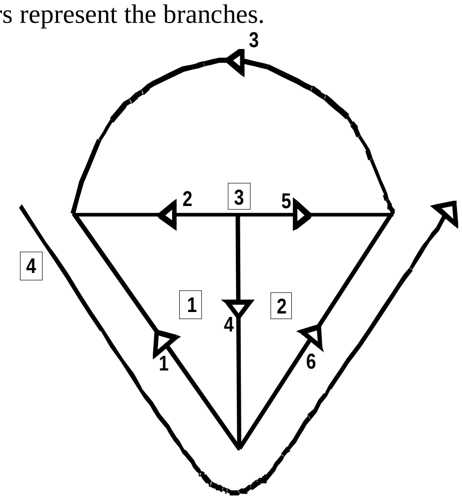

Fig. 3.2 An electrical network having four loops and six branches

$M = [m_{jk}]$, where

$$m_{jk} = \begin{cases} +1, if\ branch\ k\ is\ in\ mesh\ (j) and\ has\ same\ orientation. \\ -1, \quad if\ branch\ k\ is\ in\ mesh\ (j) and\ has\ opposite\ orientation \\ 0, \quad if\ branch\ k\ is\ not\ in\ mesh\ (j) \end{cases}$$

We can see in the figure 3.2 that, the meshes are numbered and directed in an arbitrary fashion.
Hence, M in (3) is called the mesh incidence matrix of the electrical network in the figure 3.2

$$M = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(3)}$$

Having seen that given any network, one can by analysis obtain the nodal and mesh incidence matrix of the network. We shall see that given any nodal incidence matrix one can critically draw the network.

Problem 3.1
Draw the electrical network whose nodal incidence matrix or the matrix representation is given in (4):

$$\begin{bmatrix} -1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix} \qquad \textbf{(4)}$$

Solution: The matrix (4), represent a nodal incidence matrix A suppose, with six branches since there are six columns in the matrix. Where the first, second and third row are the three different nodes. The entries of the matrix is defined by

$$a_{JK} = \begin{cases} +1, \quad if\ branch\ k\ leaves\ node\ (j) \\ -1, \quad if\ branch\ k\ enters\ node\ (j) \\ 0, \quad if\ branch\ k\ does\ not\ touch\ node\ (j) \end{cases}$$

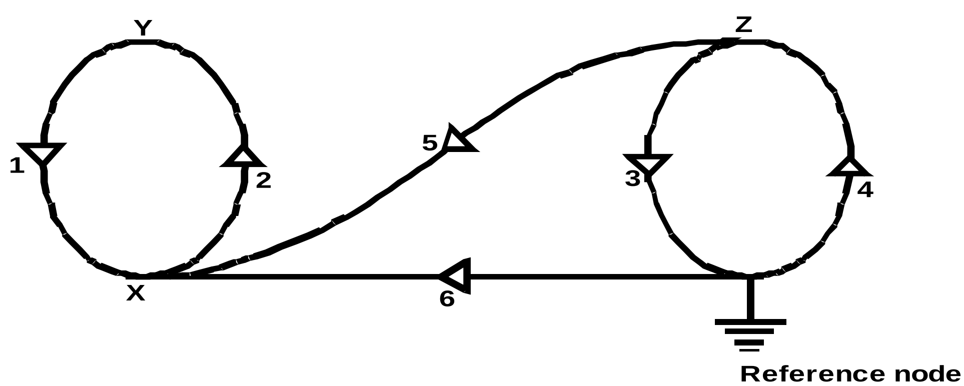

Fig 3.3 A diagram of electrical network in an open space

The figure 3.3 shows an electrical network with six branches (connections) and four nodes. One node is the reference node (ground node, whose voltage is zero).

4.0 Models of electric circuits

Most people think of electricity as something that flows through wires. In fact, it is usually convenient to understand electricity as electrons flowing through wires. When we talk of something flowing, we naturally believe that there is pressure behind the flow and the quantity of substance flowing. For electrical circuits, the pressure behind the electrons is measure in amperes or amps. For simplicity sake, we shall consider only direct current (dc) circuits, circuits in which the electricity travels in one direction in each wire.

The basic laws that govern the current flow are stated below.

Ohm's law: the voltage drop across a resistor is the product of the currents and the resistance. V= IR where V is the voltage, I is the current and R is the resistance.

Kirchhoff's first law: the sum of the currents flowing into a node is equal to the sum of the current flowing out.

Kirchhoff's second law: the algebraic sum of the voltage drops around a closed loop is equal to the total voltage in the loop.

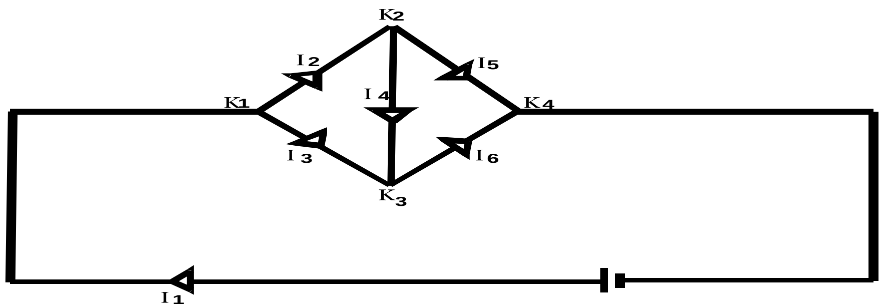

Fig 4.1 **A diagram of an electrical circuit**

The electric circuit in Fig 4.1 has four nodes (junctions) ($k_1, k_2, k_3, and\ k_4$) places where many wires meet together. There are also six branches with currentsI_1, $I_2, \ldots, I_6$ and one source of electricity in the branch from $k_4$to k_1. Each branch of this circuit has been arbitrary assigned an arrow indicating a direction of flow. The actual direction of flow will be given by the sign of the current in that branch. A positive current will flow in the opposite direction. An ammeter is used to measure both the direction and the amount of current flowing in each wire of this circuit. If the current in all the wires that meet together at a junction are added, it is found that the sum is zero. This expresses the fact that the substance (the electrons), which is flowing is not being created or destroyed at the junction. It shows that, what goes in must be equal to what comes out. This is one of the two basic laws regarding electric circuits which were first formulated by G. R. Kirchhoff in 1845.

We see that at k_1 we have I_1 amps flowing in and I_1+ I_2 amps flowing out. By Kirchhoff's law I_1- I_2 -I_3 = 0.

This is a linear equation with currents as variables. There will be one equation for each node (junction). Applying the law at the four nodes we get the following systems of equations.

I_1- I_2 -I_3 = 0 (node k_1)

I_2- I_4 -I_5 = 0 (node k_2)

I_3+I_4 -I_6 = 0 (node k_3)

-I_1+I_5+I_6 = 0 (node k_4)

Since, Kirchhoff's second law states that the sum of the voltage drops around any closed circuit must be zero. In our more intuitive language this simply means that the sum of the pressures around a circular path must be zero. In essence this law prohibits perpetual flow of electricity unless there is a source for electricity.

Gauss elimination method used in electrical networks

Gauss elimination is a method of solving linear systems of equations. It is a process of reducing the matrix coefficients of the linear system to a "triangular form" from which we shall then readily obtain the values of the unknowns by ''back substitution".

Problem 4.1. In the electrical circuit in Fig 4.2, obtain the equations for the currents and hence solve using Gaussian elimination approach.

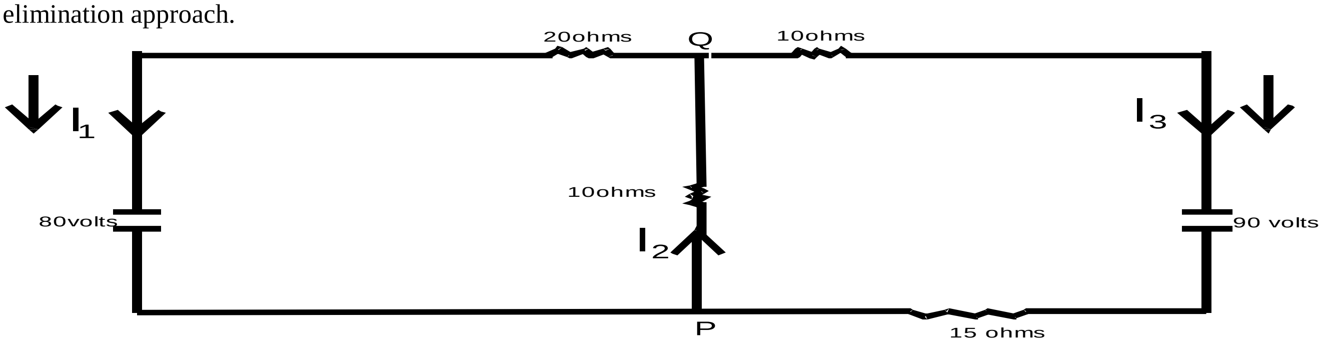

Fig 4.2 A typical diagram of an electrical circuit

To obtain the equations, we use Kirchhoff's current law (KCL) and Kirchhoff's voltage law (KVL).
Node P: $i_1 - i_2 + i_3 = 0$
Node Q: $-i_1 + i_2 - i_3 = 0$ **(5)**
Right loop: $10i_1 + 25i_3 = 90$
Left loop: $20i_1 + 10i_2 = 80$
Now for convenience purpose, let $i_1 = x_1,\ i_2 = x_2,\ i_3 = x_3$. Thus **(5)** can be written as

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_1 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80 \end{aligned} \qquad \textbf{(6)}$$

The augmented matrix of **(6)** is

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{bmatrix}$$

Which is reduced to "triangular" system B by Gaussian elimination

$$B = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Back substitution: working backward from the last to the first equation of this triangular system, we readily obtain $x_3, x_2,\ and\ x_1$
$-95x_3 = -90 \rightarrow x_3 = i_3 = 2$ amperes.
$10x_2 + 25x_3 = 90$, $x_2 = \frac{1}{10}(90 - 25x_3) = i_2 = 4$ amperes
$x_1 - x_2 + x_3 = 0,\ x_1 = i_1 = 2$ amperes.
This is the answer to the problem and the solution is unique

5.0 Conclusion

Linear algebra offers a general mathematical tool for solving system of linear equations. In this paper, we make an extension by constructing an electrical network given the nodal incidence matrix. We also solve a practical example of an electric circuit, whose solution is unique.

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