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Applications of Fourier Transform To Solutions of a First Order Linear Differential Equation

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Abstract

In this paper, we showed how Fourier transform is applied to obtain a solution of differential equation of first order linear system through the method of convolution formula and transform products.

Keywords: Fourier Transform, First order Differential Equation, Initial value Problem, Convolution, Sine Function.

1.0 Introduction

Many linear boundary value and Initial value problems in applied mathematics, Mathematical Physics and Engineering Science can be effectively solved by the use of the Fourier transform (the Fourier Cosine transform or the Fourier sine transform). These transforms are very useful for solving differential equations for the following reasons.

First, these equations are replaced by simple algebraic equations which enable us to find the solution ofthe solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution.

Secondly, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green's function.

Finally, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary valued and initial value problems [1].

Suppose that$f(x)$ is a periodic function of period 2π that can be represented by a trigonometric series of the form [2]

$$f(x) = a_o + \sum_{n=1}^{N}(a_n cosnx + b_n sinnx) \qquad (1.0)$$

The series (1.0) is assumed to converge and has $f(x)$ as its sum. Given such a function$f(x)$, he showed the determination of a_n and b_n of the corresponding series (1.0), that is,

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosmx\, dx, m = 1, 2, \ldots$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinmx\, dx, m = 1, 2, \ldots$$

The fourier transform provides a representation of functions defined over an infinite interval, and having no particular periodicity in terms of a superposition of sinusoidal functions. It may thus be considered as a generalization of the Fourier series representation of periodic functions. Since fourier transforms are often used to represent time-varying functions, we shall present much of our discussion in terms of$f(t)$, rather than $f(x)$, although in some spatial examples $f(x)$ will be more natural notation and we shall use it as appropriate. Our only requirement on $f(t)$ will be that $\int_{-\infty}^{\infty}|f(t)|\, dt$ is finite[3]

In this work, our objective is to apply Fourier transform to obtain a solution of differential equation of first order linear differential system of the form

$$\dot{x} = f(x,t); x(t_o) = x_o$$

and then convert the expressions involving derivatives of unknown functions to simpler algebraic expression.

2.0 Definitions and Theorems on Our Concept

2.1 FOURIER TRANSFORMS: If $k(t,s) = \frac{e^{-ist}}{\sqrt{2\pi}}$,$a = -\infty$ and $b = \infty$ (we will let I:=F) the transformation $I\{f(t)\} = \int_a^b k(t,s)f(t)dt = \hat{f}(s)$ becomes the Fourier transform $(i^2 = -1)$. hence, the fourier transform of f(t) is defined by $F\{f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ist}\hat{f}(s)ds = F^{-1}[F\{f(t)\}] = f(t)$ See [4]

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2.2 Piecewise Continuous Function:
Let $f(x)$ Be Defined On $[a,b]$, Except Possibly At Finitely Many Points. Then f is piecewise continuous on $[a,b]$ if;
a. f is continuous on [a,b] except perhaps at finitely many points.
b. Both $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist and are finite.
c. If x_0 is in (a,b) and f is not continuous at x_0, then $\lim_{x\to x_0^+} f(x)$ and$\lim_{x\to x_0^-} f(x)$ exist and are finite. See pg. 593 of [5]
d. Both $\lim_{x\to b^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist and are finite.

2.3 Sinc Function:
The sinc (pronounced "sink") function on $(-\infty,\infty)$ is given by the formula,

$$sinc(x) = \frac{\sin x}{x}$$

While this formula is indeterminate at $x \equiv 0$, we see that using L'H$\hat{o}$spital's rule,

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{{}^{d}/_{dx}(\sin x)}{{}^{d}/_{dx}(x)} = \lim_{x\to 0}\frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1$$

See pg. 16 of [6]

2.4 Convolution Theorem On Fourier Transform

The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transform, i.e $F[f(x) * gx=Ffx.F[gx]$. See pg. 951 of [7]

In using the Fourier transform to solve differential equations, we need an expression relating the transform f' to that of f. The following theorems provide such relationship for derivatives of any order and are called the operational rule for the Fourier transform. A similar issue arises for any integral transform when it is to be used in connection with differential equations [5]

THEOREM 2.5: Let n be a positive integer and $let\ f^{(n-1)}$denote the $n-1$derivative of f.Suppose$f^{(n-1)}$ is continuous and $f^{(n)}$ is piecewise continuous on each interval$[-\infty,\infty]$. Suppose $\int_{-\infty}^{\infty}\left|f^{(n-1)}(t)\right|dt$converges.
Suppose, $\lim_{t\to\infty} f^{(k)}(t) = \lim_{t\to-\infty} f^{(k)}(t) = 0$
For $k = 0,1,2,\dots,n-1$.Then, $F\left[f^{(n)}(t)\right](iw) = (iw)^n \hat{f}(w)$
For the proof of the above, see [8] and pg. 653 of [5]

3.0 Methodology

We will now use the method of Convolution and Transform of products of $F^{-1}\{FG\}/t$ by starting to derive the convolution formula to which we will use to solve the initial value problem of first order linear differential equations of the form

$$\dot{x} = f(x,t); x(x_o) = x_o$$

3.1 Derivation Of The Convolution Formula
We will derive the convolution formula by attempting to evaluate $f = F^{-1}\{FG\}/t$ in terms of f and g (where as usual, $f = F^{-1}\{F\} and\ g = F^{-1}\{G\}$. For this derivation, we will assume both f and G are in A so that we can use the integral formula for their transforms.

$$F(w) = \int_{-\infty}^{\infty} f(s)e^{-i2\pi sw}ds\ and\ g(T) = \int_{-\infty}^{\infty} G(s)e^{-i2\pi wt}dw \quad (3.1)$$

Assuming that the product FG is also absolutely integrable, we then have,

$$F^{-1}[F(w)G(w)]/_t = \int_{-\infty}^{\infty} f(w)G(w)\,e^{-i2\pi wt}dw$$

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(w)G(w)\right]G(w)e^{-i2\pi wt}dw$$

thus,

$$F^{-1}[F(w)G(w)]/_t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)G(w)\,e^{-i2\pi w(t-s)}dsdw \quad (3.2)$$

Let us further assume that the order of integration of this last double integral can be interchanged.
Then,

$$F^{-1}[F(w)G(w)]/_t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)G(w)\,e^{-i2\pi w(t-s)}dwds$$

$$= \int_{-\infty}^{\infty} f(s)\left[\int_{-\infty}^{\infty} G(w)\,e^{-i2\pi w(t-s)}dw\right]ds \quad (3.3)$$

But the inner integral in the last line is just the integral $g(T)$

with $T = t - s$. So the equation (3.3) reduces to

$$F^{-1}[F(w)G(w)]/_t = \int_{-\infty}^{\infty} f(s)g(t-s)ds \qquad (3.4)$$

Therefore, equation (3.4) is known as the convolution formula. See pg. 371-372 of [6]

Illustration

Let us try to solve the differential equation

$$\frac{dy}{dt} + 3y = pulse,(t) \qquad (3.5)$$

Solution

$$F\left\{\frac{dy}{dt} + 3y\right\}/_w = F\{pulse,(t)\}/_w \qquad (3.6)$$

$$\Rightarrow F\left\{\frac{dy}{dt}\right\}/_w + F\{3y\}/_w = 2sinc(2\pi w)$$

$$\Rightarrow F\left\{\frac{dy}{dt}\right\}/_w + 3F\{y\}/_w = 2sinc(2\pi w)$$

$$\Rightarrow F\left\{\frac{dy}{dt}\right\}/_w + 3F\{y\}/_w = 2sinc(2\pi w) \qquad (3.7)$$

$$\Rightarrow i2\pi wY(w) + 3Y(w) = 2sinc(2\pi w) \qquad (3.8)$$

The last equation i.e, equation (3.8) is the simple algebraic equation for Y. by simple algebra,

$$[i2\pi w + 3]Y(w) = 2sinc(2\pi w) \qquad (3.9)$$

$$Y(w) = \frac{2sinc(2\pi w)}{3 + i2\pi w}$$

Then since Y is classically transformable, we can obtain the corresponding solution to the differential equation by taking the inverse transform.

$$y(t) = F^{-1}[Y]/_t = F^{-1}\left[\frac{2sinc(2\pi w)}{3+i2\pi w}\right]/_t \qquad (3.10)$$

See pg 335-336 of [6]

4.0 Main Results

From our equation (3.10), since the sinc function is continuous and $3 + i2\pi w$ is never zero for real values of w. it should be clear that Y is continuous. We show that $Y(w)$ and $wY(w)$ are classically transformable by showing that,

i. $Y(w)$ is absolutely integrable and
ii. $wY(w)$is the product of a known classically transformable function with a sine function.

Since we have solved a first order linear differential equation of the form

$\dot{x} = f(x,t); x(t_0) = x_0$ and convert it into simple algebraic equation then, we will use the boundary value or the interval of integration of the form $0 < t < 1$ and with the help of convolution formula which states that

$$F^{-1}[F(w)G(w)]/_t = \int_{-\infty}^{\infty} f(s)g(t-s)ds$$

Then from equation (3.10) which is

$$y(t) = F^{-1}[Y]/_t = F^{-1}\left[\frac{2sinc(2\pi w)}{3 + i2\pi w}\right]/_t$$

Let $F(w) = 2sinc(2\pi w) and\ G(w) = \frac{1}{3+i2\pi w}$ whose inverse transforms are;

$$f(t) = F^{-1}[F]/_t = pulse,(t) and$$
$$g(t) = F^{-1}[G]/_t = e^{-3t} step\ (t)$$

From the selected Fourier Transform table in pg. 312 of [6]

So that,

$$\int_{-\infty}^{\infty} \frac{2sinc(2\pi w)}{3 + i2\pi w} dw = \int_{-\infty}^{\infty} 2sinc(2\pi w)\frac{1}{3 + i2\pi w} dw$$

$$= \int_{-\infty}^{\infty} F[pulse,(t)]/_t \frac{1}{3 + i2\pi w} dw$$

$$= \int_{-\infty}^{\infty} pulse,(t) F^{-1}\left[\frac{1}{3 + i2\pi w}\right]/_t dt$$

$$= \int_{-\infty}^{\infty} e^{-3t} step\ (t) = \int_0^1 e^{-3t} dt$$

$$= \left[\frac{-e^{-3t}}{3}\right]_0^1 = \frac{-e^{-3}}{3} - \left(\frac{-e^{0}}{3}\right)$$
$$= \frac{-e^{-3}}{3} + \frac{1}{3}$$
$$= \frac{1}{3} - \frac{-e^{-3}}{3}$$

Therefore, the main result which is to obtain the corresponding solution to the differential equation for which we have taken the inverse Fourier transform and then applied the method of convolution formula, is given as

$$\int_{-\infty}^{\infty} \frac{2sinc(2\pi w)}{3 + i2\pi w} dw = \frac{1}{3}(1 - e^{-3})$$

5.0 Discussion

In this work, we were able to observe the following:

i. We achieved our objective by applying Fourier transform to obtain a solution of first order linear differential equation of the form $\dot{x} = f(x,t); x(t_0) = x_0$and then converting the expression involving derivatives into a simpler algebraic equation.

ii. Our problem was solved through the method of convolution formula and transforms of products from which we derived the convolution formula.

iii. The method used in our work is helpful in verifying the Fourier inverse of the Fourier transform of first order linear differential equations of the form

$$\dot{x} = f(x,t); x(t_0) = x_0$$

after converting the expressions into simple algebraic equations.

iv. Both F and G are classically transformable and absolutely integrablei.e we can transformed the products of F and g and integrate them in an infinite interval of $[-\infty, \infty]$ as we have derived.

v. The product FG is absolutely integrable which we have shown.

vi. The order of integration in $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)G(w)\, e^{-i2\pi w(t-s)} dwds$ can be interchanged as we have seen in our methodology where we interchange our integration.

Our suggestion towards this work is that one can pick any method of Fourier transform rather than method of convolution formula and obtain more solutions of a first order linear differential equation of the form;
$\dot{x} = f(x,t); x(t_0) = x_0$which is Appliedin Fourier transform to obtain solutions of an initial valued problem of first order linear differential equation.

6.0 Conclusion

We have seen the few applications or derivation of Fourier transform. The uses of the Fourier transform cannot be over emphasized.Most periodic phenomena such as rotating machines, sound waves, signals, statistical time series, vibrating wave under the earth crust, etc. the Fourier transform methods can be applied to the above listed problems.We have also seen one of the reasons why Fourier transform is so important in many applications which is by converting expressions involving derivatives of unknown functions tosimpler algebraic equation thereby verifying a solution to a first order linear differential equation of the form

$\dot{x} = f(x,t); x(t_0) = x_0$.

Finally,the method of convolution formula and transforms of product is also important especially in verifying the inverse Fourier transform of a solution to a differential equation.

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