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New Travelling Wave Solution of The Korteweg De Vries Equation By (G'/G) - Expansion Method And Liu's Theorem.

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Abstract

Exact hyperbolic, trigonometric and rational function travelling wave solutions to the Korteweg De Vries (KDV) equation using the (G'/G) expansion method are presented in this paper. New travelling wave solutions to the KDV equation were obtained with Liu's theorem. The solutions obtained were verified by putting them back into the equation with the aid of Mathematica. This shows that the (G'/G) expansion method is a powerful and effective tool for obtaining exact solutions to nonlinear partial differential equations in physics, mathematics and other applications.

Keywords: (G'/G) expansion method, KDV equation, travelling wave solutions, Solitons, nonlinear partial differential equations**.**

1.0 Introduction

Most physical phenomena are described by partial differential equations that are nonlinear in nature. These nonlinear partial differential equations (NLPDE) appear in many fields such as Hydrodynamics, Engineering, Quantum field theory, Optics, Plasma physics, Biology, Fluid mechanics, Solid-state physics, etc. [1]. In Soliton theory, the study of exact solutions to these nonlinear equations plays a very germane role, as they provide much information about the physical models they describe.

Various powerful methods have been employed to construct exact solutions to nonlinear partial differential equations. These methods include the inverse scattering transform [2], the Backlund transform [3-4], the Darboux transform [5], the Hirota bilinear method [6], the tanh-function method [7-8}, the sine-cosine method [9], the exp-function method [10], the generalized Riccati equation [11], the Homogenous balance method [12], etc.

Recently a new method called the (G'/G) expansion method was introduced by Wang et al. [13] to construct exact solution to nonlinear partial differential equations.

The Korteweg de Vries equation (KDV) which is a non-linear PDE of third order has been of interest since 150 years ago. More recently, this equation has been found to describe wave phenomena in Plasma physics, anharmonic crystals, etc. The KDV is a non-linear equation which has soliton solutions. The Korteweg de Vries equation is of the form

$$\frac{\partial u(x,t)}{\partial t}+pu(x,t)\frac{\partial u(x,t)}{\partial x}+q\frac{\partial^3 u(x,t)}{\partial x^3}=0$$

KDV is non-linear because of the product shown in the second summand and is of third order because of the third derivative in the third summand. The aim of this paper is to construct travelling wave solution to the Korteweg de Vries equation by using the (G'/G) expansion method.

1.0 Description of the (G'/G) expansion method.

Here, we provide a brief explanation of the (G'/G) expansion method for finding travelling wave solutions of nonlinear partial differential equations. Consider a nonlinear equation of two independent variables x and t of the form:

$$P(u,u_t,u_x,u_{tt},u_{xx},u_{xt},\dots)=0 \qquad (2)$$

P is a polynomial in $u(x,t)$ and its derivatives with respect to x and t. To begin, we transform equation (2) into a nonlinear ordinary differential equation by introducing the variable ξ given by:

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$$u(x,t) = u(\xi) \qquad \xi = x - ct \tag{3}$$

Where c is a constant and equation (2) reduces to a nonlinear ordinary differential equation of the form

$$Q(u, u', u'', u''' \ldots) = 0 \tag{4}$$

The method assumes that the solution to equation (2) can be expressed as a polynomial in (G'/G)

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{G'}{G}\right)^i \qquad \alpha_m \neq 0 \tag{5}$$

Where $\alpha_0, \alpha_1, \ldots, \alpha_m$ are constants to be determined and $G = G(\xi)$ is a solution of the linear ordinary differential equation of the form

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0 \tag{6}$$

λ and μ are arbitrary constants. The general solution of (6) gives [14]:

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left[\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right] - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0 \ \ Hyperbolic \\ \frac{\sqrt{4\mu - \lambda^2}}{2}\left[\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right] - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0 \ \ Trigonometric \\ \frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0 \ \ Rational \end{cases}$$

Equation (4) is integrated as long as all the terms contain derivatives, where integration constants are considered to be zero. To determine m, we consider the homogenous balance between the highest order derivative and the highest order nonlinear term(s).

Substitute equation (5) with the determined value of m into equation (4), and collect all terms with the same order of (G'/G) together. If the coefficients of $(G'/G)^i$ vanish separately, we have a set of algebraic equations in $\alpha_0, \alpha_1, \ldots, \alpha_m, c, \lambda$ and μ that is solved with the aid of Mathematica.

Finally, substituting $\alpha_0, \alpha_1, \ldots, \alpha_m, c$ and the general solution to equation (6) into equation (5) yield the travelling wave solution of equation (2).

2.0 Application

In this section, we apply the (G'/G) expansion method to construct travelling wave solution of the Korteweg de Vries equation. Consider the KDV equation (1) which can be written as:

$$u_t + puu_x + qu_{xxx} = 0 \tag{7}$$

We make the transformation $(x,t) = u(\xi)$, $\xi = x - ct$.

$$\frac{\partial u}{\partial t} = \frac{du}{d\xi}\cdot\frac{\partial \xi}{\partial t} = -c\frac{du}{d\xi}$$

$$\frac{\partial u}{\partial x} = \frac{du}{d\xi}\cdot\frac{\partial \xi}{\partial x} = \frac{du}{d\xi}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right)$$

$$= \frac{d}{d\xi}\cdot\frac{\partial \xi}{\partial x}\left[\frac{d}{d\xi}\cdot\frac{\partial \xi}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)\right] = \frac{d}{d\xi}\left[\frac{d}{d\xi}\left(\frac{\partial u}{\partial \xi}\right)\right] = \frac{d^3 u}{d\xi^3}$$

Equation (7) becomes

$$-cu' + puu' + qu''' = 0 \tag{8}$$

Integrating equation (8) once with respect to ξ and setting the integration constant to zero yields

$$-cu + p\frac{u^2}{2} + qu'' = 0 \tag{9}$$

To get m, we balance the highest order derivative u'' and the highest order nonlinear term u^2.

$$2m = m + 2 \implies m = 2$$

Then equation (5) becomes

$$u(\xi) = \alpha_2 \left(\frac{G'}{G}\right)^2 + \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0 \qquad \text{where } \alpha_2 \neq 0 \tag{10}$$

Where α_0, α_1 and α_2 are constants to be determined later. From (10)

$$u'(\xi) = -2\alpha_2 \left(\frac{G'}{G}\right)^3 + (-2\alpha_2\lambda - \alpha_1)\left(\frac{G'}{G}\right)^2 + (-2\alpha_2\mu - \alpha_1\lambda)\left(\frac{G'}{G}\right) - \alpha_1\mu$$

$$u''(\xi) = 6\alpha_2 \left(\frac{G'}{G}\right)^4 + (10\alpha_2\lambda + 2\alpha_1)\left(\frac{G'}{G}\right)^3 + (8\alpha_2\mu + 4\alpha_2\lambda^2 + 3\alpha_1\lambda)\left(\frac{G'}{G}\right)^2 + (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)\left(\frac{G'}{G}\right) + 2\alpha_2\mu^2 + \alpha_1\lambda\mu$$

Substituting equation (10) and its derivatives into equation (9) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together yields a simultaneous set of nonlinear algebraic equations as follows:

$$\left(\frac{G'}{G}\right)^0: \quad 4\alpha_2\mu^2 q + 2\alpha_1\lambda\mu q + {\alpha_0}^2 p - 2\alpha_0 c = 0$$

$$\left(\frac{G'}{G}\right)^1: \quad 6\alpha_2\lambda\mu q + 2\alpha_1\mu q + \alpha_1\lambda^2 q + \alpha_0\alpha_1 p - \alpha_1 c = 0$$

$$\left(\frac{G'}{G}\right)^2: \quad 16\alpha_2\mu q + 6\alpha_1\lambda q + 8\alpha_2\lambda^2 q + 2\alpha_0\alpha_2 p + {\alpha_1}^2 p - 2\alpha_2 c = 0$$

$$\left(\frac{G'}{G}\right)^3: \quad 2\alpha_1 q + 10\alpha_2\lambda q + \alpha_1\alpha_2 p = 0$$

$$\left(\frac{G'}{G}\right)^4: \quad 12\alpha_2 q + {\alpha_2}^2 p = 0$$

Solving this algebraic system of equation with the aid of Mathematica yields two different sets of solution:

$$\text{Case 1:} \quad \alpha_2 = \frac{-12q}{p} \qquad \alpha_1 = \frac{-12q\lambda}{p} \qquad \alpha_0 = \frac{-12q\mu}{p} \qquad c = q(\lambda^2 - 4\mu)$$

$$\text{Case 2:} \quad \alpha_2 = \frac{-12q}{p} \qquad \alpha_1 = \frac{-12q\lambda}{p} \qquad \alpha_0 = \frac{-2q(\lambda^2 + 2\mu)}{p} \qquad c = -q(\lambda^2 - 4\mu)$$

Substituting the different sets of solution to the algebraic equation and the general solution to equation (6) into equation (10), we obtain three types of travelling wave solutions of the Korteweg de Vries equation.

<u>Case 1</u>

When $\lambda^2 - 4\mu > 0$, we obtain a hyperbolic function solution:

$$u(\xi) = \frac{-12q}{p}\left(A\left[\frac{C_1\sinh(A\xi) + C_2\cosh(A\xi)}{C_1\cosh(A\xi) + C_2\sinh(A\xi)}\right] - \frac{\lambda}{2}\right)^2 - \frac{12q\lambda}{p}\left(A\left[\frac{C_1\sinh(A\xi) + C_2\cosh(A\xi)}{C_1\cosh(A\xi) + C_2\sinh(A\xi)}\right] - \frac{\lambda}{2}\right) - \frac{12q\mu}{p}$$

Where $A = \frac{\sqrt{\lambda^2 - 4\mu}}{2}$ and $\xi = x - q(\lambda^2 - 4\mu)t$

$$u(\xi) = \frac{12q}{p}A^2\left[1 - \left(\frac{C_1\sinh(A\xi) + C_2\cosh(A\xi)}{C_1\cosh(A\xi) + C_2\sinh(A\xi)}\right)^2\right] \tag{11}$$

If we set $C_1 = 0$ and $C_2 \neq 0$ in equation (11), we obtain

$$u(\xi) = -\frac{3q}{p}(\lambda^2 - 4\mu)\left[\operatorname{csch}^2\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}[x - q(\lambda^2 - 4\mu)t]\right)\right] \tag{12}$$

If we set $C_1 \neq 0$ and $C_2 = 0$ in equation (11), we obtain

$$u(\xi) = \frac{3q}{p}(\lambda^2 - 4\mu)\left[\operatorname{sech}^2\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}[x - q(\lambda^2 - 4\mu)t]\right)\right] \tag{13}$$

When $\lambda^2 - 4\mu < 0$, we obtain a trigonometric function solution:

$$u(\xi) = \frac{-12q}{p}\left(B\left[\frac{-C_1\sin(B\xi) + C_2\cos(B\xi)}{C_1\cos(B\xi) + C_2\sin(B\xi)}\right] - \frac{\lambda}{2}\right)^2 - \frac{12q\lambda}{p}\left(B\left[\frac{-C_1\sin(B\xi) + C_2\cos(B\xi)}{C_1\cos(B\xi) + C_2\sin(B\xi)}\right] - \frac{\lambda}{2}\right) - \frac{12q\mu}{p}$$

Where $B = \frac{\sqrt{4\mu-\lambda^2}}{2}$ and $\xi = x + q(4\mu - \lambda^2)t$

$$u(\xi) = -\frac{12q}{p}B^2\left[1 + \left(\frac{-C_1\sin(B\xi) + C_2\cos(B\xi)}{C_1\cos(B\xi) + C_2\sin(B\xi)}\right)^2\right] \tag{14}$$

If we set $C_1 = 0$ and $C_2 \neq 0$ in equation (14), we obtain

$$u(\xi) = -\frac{3q}{p}(4\mu - \lambda^2)\left[\csc^2\left(\frac{\sqrt{4\mu - \lambda^2}}{2}[x + q(4\mu - \lambda^2)t]\right)\right] \tag{15}$$

If we set $C_1 \neq 0$ and $C_2 = 0$ in equation (14), we obtain

$$u(\xi) = -\frac{3q}{p}(4\mu - \lambda^2)\left[\sec^2\left(\frac{\sqrt{4\mu - \lambda^2}}{2}[x + q(4\mu - \lambda^2)t]\right)\right] \tag{16}$$

When $\lambda^2 - 4\mu = 0$, we obtain a rational function solution:

$$u(\xi) = \frac{-12q}{p}\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right)^2 - \frac{12q\lambda}{p}\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right) - \frac{12q\mu}{p}$$

Where $\xi = x$

$$u(\xi) = -\frac{12q}{p}\left(\frac{C_2}{C_1 + C_2 x}\right)^2 \tag{17}$$

<u>Case 2</u>

When $\lambda^2 - 4\mu > 0$, we obtain a hyperbolic function solution:

$$u(\xi) = \frac{-12q}{p}\left(A\left[\frac{C_1\sinh(A\xi) + C_2\cosh(A\xi)}{C_1\cosh(A\xi) + C_2\sinh(A\xi)}\right] - \frac{\lambda}{2}\right)^2 - \frac{12q\lambda}{p}\left(A\left[\frac{C_1\sinh(A\xi) + C_2\cosh(A\xi)}{C_1\cosh(A\xi) + C_2\sinh(A\xi)}\right] - \frac{\lambda}{2}\right) - \frac{2q(\lambda^2 + 2\mu)}{p}$$

Where $A = \frac{\sqrt{\lambda^2-4\mu}}{2}$ and $\xi = x + q(\lambda^2 - 4\mu)t$

$$u(\xi) = \frac{12q}{p}A^2\left[\frac{1}{3} - \left(\frac{C_1\sinh(A\xi) + C_2\cosh(A\xi)}{C_1\cosh(A\xi) + C_2\sinh(A\xi)}\right)^2\right] \tag{18}$$

If we set $C_1 = 0$ and $C_2 \neq 0$ in equation (18), we obtain

$$u(\xi) = -\frac{3q}{p}(\lambda^2 - 4\mu)\left[\coth^2\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}[x + q(\lambda^2 - 4\mu)t]\right)\right] + \frac{q}{p}(\lambda^2 - 4\mu) \tag{19}$$

If we set $C_1 \neq 0$ and $C_2 = 0$ in equation (18), we obtain

$$u(\xi) = -\frac{3q}{p}(\lambda^2 - 4\mu)\left[\tanh^2\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}[x + q(\lambda^2 - 4\mu)t]\right)\right] + \frac{q}{p}(\lambda^2 - 4\mu) \tag{20}$$

When $\lambda^2 - 4\mu < 0$, we obtain a trigonometric function solution:

$$u(\xi) = \frac{-12q}{p}\left(B\left[\frac{-C_1\sin(B\xi) + C_2\cos(B\xi)}{C_1\cos(B\xi) + C_2\sin(B\xi)}\right] - \frac{\lambda}{2}\right)^2 - \frac{12q\lambda}{p}\left(B\left[\frac{-C_1\sin(B\xi) + C_2\cos(B\xi)}{C_1\cos(B\xi) + C_2\sin(B\xi)}\right] - \frac{\lambda}{2}\right) - \frac{2q(\lambda^2 + 2\mu)}{p}$$

Where $B = \frac{\sqrt{4\mu-\lambda^2}}{2}$ and $\xi = x - q(4\mu - \lambda^2)t$

$$u(\xi) = -\frac{12q}{p}B^2\left[\frac{1}{3} + \left(\frac{-C_1\sin(B\xi) + C_2\cos(B\xi)}{C_1\cos(B\xi) + C_2\sin(B\xi)}\right)^2\right] \tag{21}$$

If we set $C_1 = 0$ and $C_2 \neq 0$ in equation (21), we obtain

$$u(\xi) = -\frac{3q}{p}(4\mu - \lambda^2)\left[\cot^2\left(\frac{\sqrt{4\mu - \lambda^2}}{2}[x - q(4\mu - \lambda^2)t]\right)\right] - \frac{q}{p}(4\mu - \lambda^2) \tag{22}$$

If we set $C_1 \neq 0$ and $C_2 = 0$ in equation (21), we obtain

$$u(\xi) = -\frac{3q}{p}(4\mu - \lambda^2)\left[\tan^2\left(\frac{\sqrt{4\mu - \lambda^2}}{2}[x - q(4\mu - \lambda^2)t]\right)\right] - \frac{q}{p}(4\mu - \lambda^2) \tag{23}$$

When $\lambda^2 - 4\mu = 0$, we obtain a rational function solution:

$$u(\xi) = \frac{-12q}{p}\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right)^2 - \frac{12q\lambda}{p}\left(\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right) - \frac{2q(\lambda^2 + 2\mu)}{p}$$

Where $\xi = x$

$$u(\xi) = -\frac{12q}{p}\left(\frac{C_2}{C_1 + C_2 x}\right)^2 \tag{24}$$

Theorem 1: Liu's theorem [15].
If equation (2) has a kink-type solution in the form

$$u(\xi) = P_k(\tanh[\psi\xi]) \tag{25}$$

Where P_k is a polynomial of degree k. Then it has a certain kink bell type solution in the form

$$u(\xi) = P_k(\tanh[2\psi\xi] \pm i\,\mathrm{sech}[2\psi\xi]) \tag{26}$$

i is the imaginary number unit.
We can apply Theorem 1 to equation (12), (13), (19) and (20) to construct new set of travelling wave solutions to the Korteweg De Vries equation.
Using Theorem 1, equation (12) becomes

$$u(\xi) = -\frac{3q}{p}(\lambda^2 - 4\mu) + \frac{3q}{p}(\lambda^2 - 4\mu)\left[\tanh\left(\sqrt{\lambda^2 - 4\mu}\,\xi\right) \pm i\,\mathrm{sech}\left(\sqrt{\lambda^2 - 4\mu}\,\xi\right)\right]^{-2} \tag{27}$$

Where $\xi = x - q(\lambda^2 - 4\mu)t$
Using Theorem 1, equation (13) becomes

$$u(\xi) = \frac{3q}{p}(\lambda^2 - 4\mu) - \frac{3q}{p}(\lambda^2 - 4\mu)\left[\tanh\left(\sqrt{\lambda^2 - 4\mu}\,\xi\right) \pm i\,\mathrm{sech}\left(\sqrt{\lambda^2 - 4\mu}\,\xi\right)\right]^{2} \tag{28}$$

Where $\xi = x - q(\lambda^2 - 4\mu)t$
Using Theorem 1, equation (19) becomes

$$u(\xi) = \frac{q}{p}(\lambda^2 - 4\mu) - \frac{3q}{p}(\lambda^2 - 4\mu)\left[\tanh\left(\sqrt{\lambda^2 - 4\mu}\,\xi\right) \pm i\,\mathrm{sech}\left(\sqrt{\lambda^2 - 4\mu}\,\xi\right)\right]^{-2} \tag{29}$$

Where $\xi = x + q(\lambda^2 - 4\mu)t$
Using Theorem 1, equation (20) becomes

$$u(\xi) = \frac{q}{p}(\lambda^2 - 4\mu) - \frac{3q}{p}(\lambda^2 - 4\mu)\left[\tanh\left(\sqrt{\lambda^2 - 4\mu}\,\xi\right) \pm i\,\mathrm{sech}\left(\sqrt{\lambda^2 - 4\mu}\,\xi\right)\right]^{2} \tag{30}$$

Where $\xi = x + q(\lambda^2 - 4\mu)t$

The travelling wave solutions of the Korteweg De Vries equation obtained using the (G'/G) expansion method for the hyperbolic, trigonometric and rational function types are presented in equations (11), (14), (17), (18), (21) and (24). When the arbitrary constants C_1 and C_2 are taken to be zero separately in equations (11), (14), (18) and (21), we derive special soliton solution presented in equations (12), (13), (15), (16), (19), (20), (22) and (23). If we set $p = 6$, $q = 1$ and $\lambda^2 - 4\mu = 2\beta$, equations (12) and (13) become identical to Brauer's solutions to the KDV equation [16].

Applying Liu's theorem to equations (12), (13), (19) and (20), we obtain new travelling wave solutions in the form of equations (27), (28), (29) and (30). All the travelling wave solutions to the KDV equation obtained were checked by putting them back into equation (1) with the aid of Mathematica.

3.0 Conclusion

Hyperbolic, trigonometric and rational function travelling wave solutions to the Korteweg De Vries equation have been obtained using the (G'/G) expansion method. Liu's theorem was also applied to obtain other travelling wave solutions. The results have been verified by putting them back into the KDV equation with Mathematica. Conclusively, the (G'/G) expansion method is a powerful tool for finding exact solution to nonlinear partial differential equations in physics, mathematics and other fields.

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