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Application of Linear Algebra in Controlling Traffic flow in a Net of One-Way Streets

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Abstract

Application of linear algebra to other branches of science, engineering and economics or elsewhere occurs via the need to solve such system of linear equation. The main aim of linear algebra is to find the most economical way of manipulating and solving such systems. In this paper, we demonstrate the use of linear algebra in controlling traffic flow in net of one-way street. This can be achieved with the help of the Kirchhoff`s current law.

Keywords: linear systems, augmented matrix, traffic flow, one-way street.

1.0 Introduction

Linear algebra is a branch of mathematics concerned with the study of the theory and application of linear systems of equations (briefly called linear systems), linear transformations and eigen-value problems as they arise from electrical networks, frameworks in mechanics, curve fitting and other optimization problems, processes in statistics and systems of differential equations are examples of applications of linear algebra. Among other applications of linear algebra, this paper deals mainly with application of linear algebra in modelling traffic flow in net of one- way street. Linear algebra entered applied mathematics more than sixty years ago and is of increasing importance in various fields, especially in science, engineering and social sciences [1].Kirchhoff's current law was used in [2, 4, 5] for the analysis of electrical circuits. Basic definitions of some terms related to matrix and linear algebra were also obtained in [3].Gauss-Jordan and Gaussian elimination is used to solve the systems of the linear equations generated [6].

Sylvester in 1948 introduced the term "matrix" which was the latin word for the womb as a name for an array of numbers [7].Fredrich Gauss in 1800 develop Gaussian elimination method and use it to solve least squares problems in celestial bodies (stars, moon, planet etc) [7].

2.0 Preliminaries

A matrix is a rectangular array of numbers or functions enclosed in a square bracket.

Determinant: To every square matrix A over a field K there is assigned a specific scalar called the determinant of A. It is usually denoted by det(A) or **/A/.**

Singular and Nonsingular matrices: A square matrix A for which the determinant /A/=0 is said to be singular or noninvertible and a square matrix

A for which the determinant /A/$\neq$ 0 is said to be nonsingular or invertible.

Double suffix notation: This indicates the position of each entry in the matrix. Example a_{34}denotes an element in the third row and fourth column.

Order: A matrix order (m$\times$n) denotes m rows and n columns.

Inverse matrix: If a matrix is an n$\times$n matrix, then an n$\times$n matrix B such that AB=BA= I_n is called an inverse of the matrix A.

Echelon matrix: a matrix A= [a_{ij}]is an echelon matrix or is said to be in echelon form, if the number of zeros preceding the first nonzero entry of a row increases row by row until only zero rows remain, that is , if there exist nonzero entries. $a_1\ j_1, a_2\ j_2$, …, $a_r\ j_r$ Where j_1 <j_2< …<j_r with the property thata_{1j} = 0 for i $\leq$ r.j< j_i and for i >r. we call$a_1 j_1$,…, $a_r j_r$ the distinguished elements of the echelon matrix A.

Rank of a matrix: let A be an m$\times$n. The maximum number of linearly independent rows (column) of A is called the row (column) rank of A. if the row rank of A= the column rank of A= r (suppose),then we say that, the rank of A is r.

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Linear equations: An expression of the form

$a_1\ x_1 + a_2\ x_2 + \ldots + a_n$ ……………………………………………….………(1)

is called a linear equation. Where a_i, b∈ℝ andthe x_i, i∈ℕare indeterminants (or unknowns or variables). The scalars a_i are called the coefficients of the x_i respectively and b is called the constant term or simply constant of the equation.

A set of values for the unknowns say $x_1 = k_1$, $x_2 = k_2$,…, $x_n = k_n$ is a solution of (1). If the statement obtained by substituting k_i for x_i $a_1\ k_1 + a_2\ k_2 + \ldots + a_n\ k_n$ = b is true. This set of values is then said to satisfy the equation. If there is no ambiguity about the position of the unknowns in the equation, then we denote this solution by simply the n-tuple U= $(k_1, k_2, \ldots, k_n)$

Linear systems of equations: A linear systems of m equations in n unknowns$x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\left.\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}a_n = b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The a_{jk} are given numbers, which are called the coefficients of the system. The b_i's are also given numbers. If the b_i's are all zero, then (2) is called a homogeneous system. If at least one b_i is not zero, then (2) is called non homogeneous system. A solution of (2) is a set of numbers $x_1, \ldots, x_n$ that satisfy all the m equations. If the system (2) is homogeneous, it has at least the trivial solution $x_1 = 0, \ldots, x_n$=0.

The system of linear equations (2) is said to be over determined if the number of unknowns is less than the number of equations.

The system of linear equations (2) is called under determined if the number of unknowns is more than the number of equations.

The system of equations (2) is said to be determined if the number of unknowns is equal to the number of equations, that is m= n. The linear system of equations (2) is said to be consistent if it has a unique solution, otherwise, the linear system is said to be inconsistent (that is has infinitely many solutions).

Coefficient Matrix, AugmentedMatrix: from the definition of matrix multiplication, we see that the m equations of (2) can be written as single vector equation AX = b, where the coefficient matrix A=$[a_{jk}]$is the m×n matrix

$$A = \begin{bmatrix} a_{11}\, a_{12}\, \ldots, a_{1n} \\ a_{21}\, a_{22}\, \ldots, a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m2}\, \ldots\ldots\ldots a_{mn} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_m \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} a_{11}\, a_{12}\, \ldots, a_{1n}\, b_1 \\ a_{21}\, a_{22}\, \ldots, a_{2n}\, b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \\ a_{m2}\, \ldots\ldots\ldots a_{mn}\, b_m \end{bmatrix}$$

is called the augmented matrix of the system (2). D is obtained by augmenting A by the column b. The matrix D determines the system (2) completely because it contains all the given numbers appearing in (2).

Kirchhoff's First Law: the sum of the currents flowing in to a node is equal to the sum of the current flowing out.

3.0 Linear Algebra Applied in Controlling of Traffic Flow

The method of electrical network analysis has application to other fields.For instance, in applying the analog of Kirchhoff"s current law (KCL) to determine the vehicle per hour (vph)ina net of one way street. This shows how solving a system of linear equations can help to solve practical problems involving traffic flow. To make things more concrete, we consider

Problem 1. The map in Fig. 1 represents traffic flow through a certain block of one-way streets in Kingston city in Australia (the numbers are the average flows in to and out of the network at peak traffic hours).

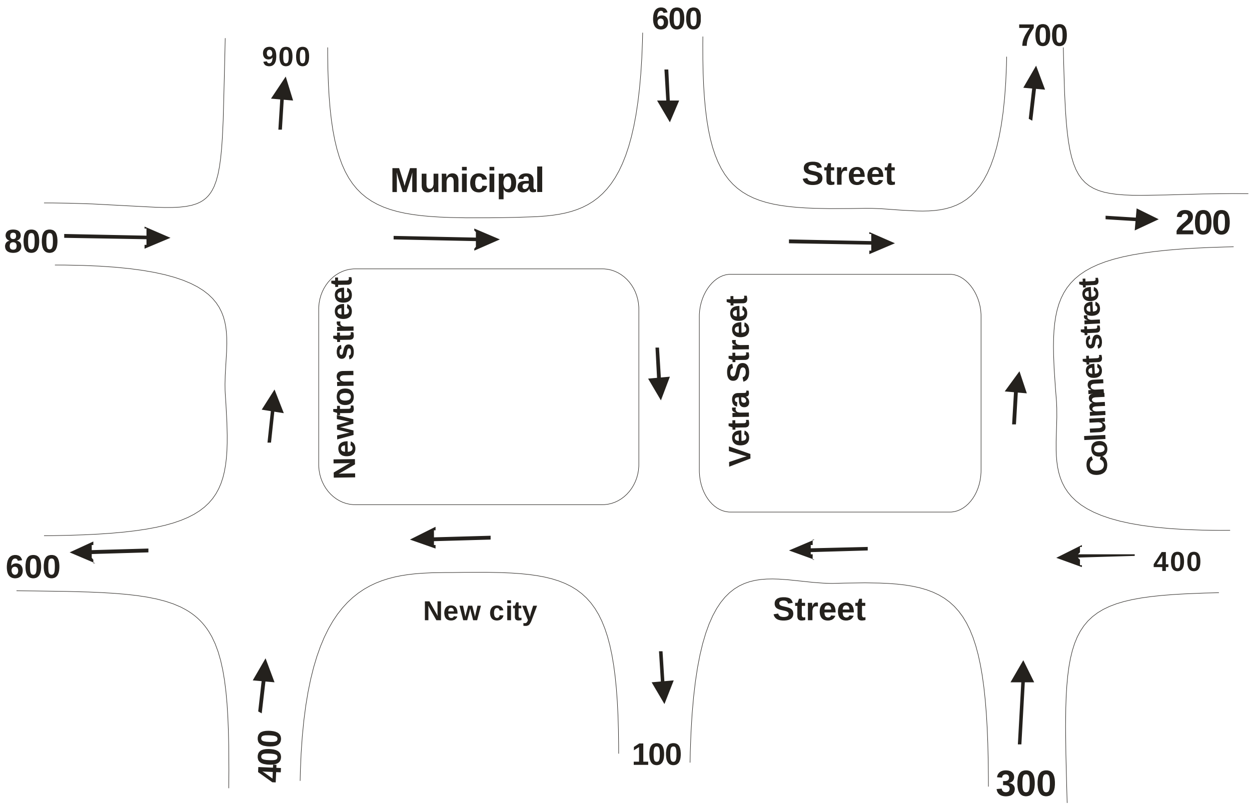

Figure 3.1

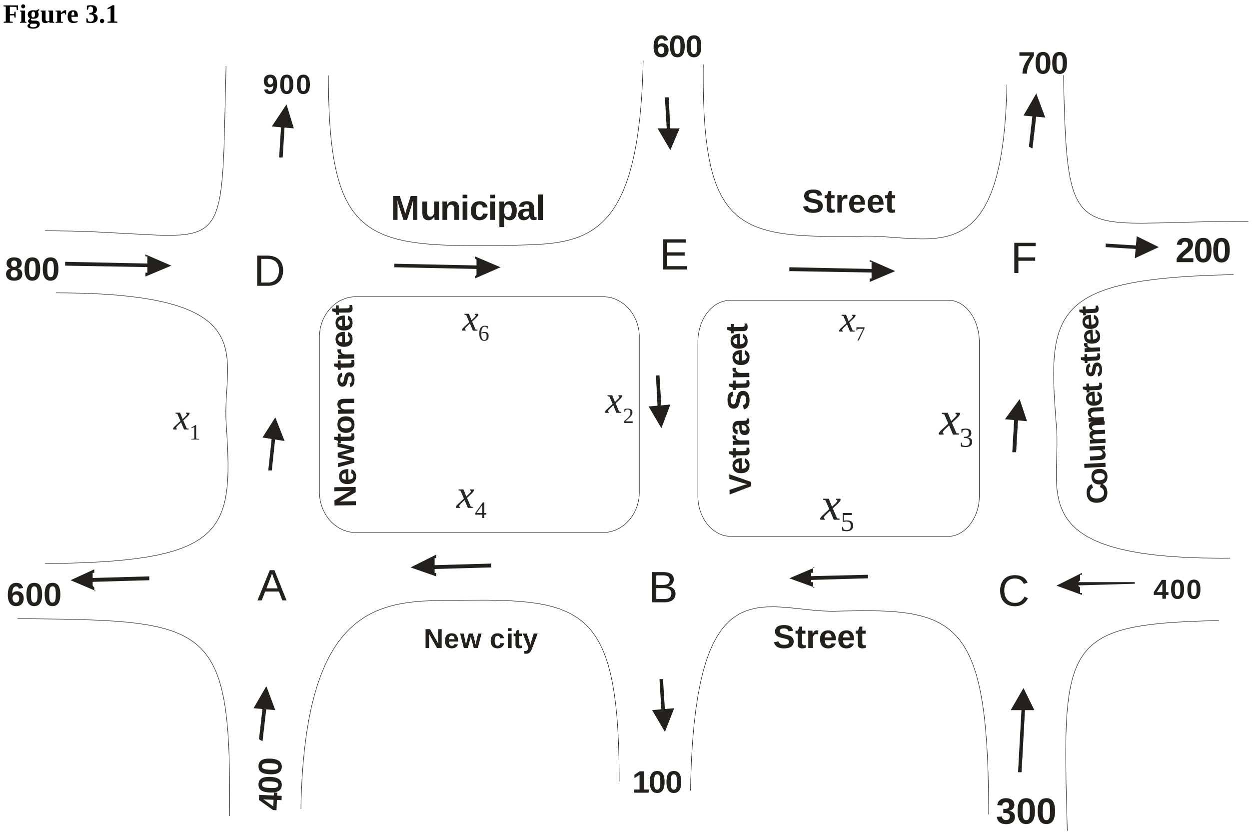

Figure 3.2 A map showing net of one-way streets with average number of traffic in a section of Kingston city.

On the map we have indicated the traffic flow in and out of each street; the units are vehicles per hour (vph). Since traffic flow varies greatly during the day, we shall assume that the numbers given represents average traffic flow at the peak period, which is approximately 4pm to 5:30pm.

Suppose that a political group is planning a rally on Newcity street between VetraStreet and Newton's street at 5:00pm on Monday. The Kingston city road marshals can to a certain extent control the traffic flow by resetting traffic lights and stationing the road marshals at certain key intersections or closing critical streets to all vehicular traffic, if traffic is slowed on New city street, it will increase on the adjacent streets. The problem becomes one of minimizing traffic flow in New city street (between vetra street and Newton's street)without causing traffic tie –up on the other streets. To solve our minimization problem, we add some labels to our map as in the figure 1.

We have labeled the six junctions A through F and have denoted the traffic flow between adjacent streets by the variable$x_1$through x_7. The problem now is to minimize x_4 subject to the constraints imposed by the problem. At the junction B, the traffic flowing out is x_4 + 100. Assuming that no traffic is backed up at B, the "in" traffic must equal the "out'' traffic. Therefore, we obtain the equation $x_2 + x_5 = x_4 + 100$ or $x_2 - x_4 + x_5 = 100$. Doing similar analysis at each junction, we obtain the following system of six equations in seven unknowns

At A: $x_1 - x_4 = -200$

At B: $x_2 - x_4 + x_5 = 100$

At C: $x_3 + x_5 = 700$

At D: $x_1 - x_6 = 100$

At E: $x_2 - x_6 + x_7 = 600$

At F: $x_3 + x_7 = 900$

We write this system in augmented form and solve it by row reduction

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 100 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 700 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 900 \end{bmatrix}$$

$R_4 \to R_4 - R_1$means new row 4 implies old row 4 minus row 1

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 100 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 300 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 900 \end{bmatrix}$$

$R_5 \to R_5 - R_2$means new row 5 implies old row 5 minus row 2

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 100 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 300 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 500 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 900 \end{bmatrix}$$

$R_6 \to R_6 - R_3$, means new row 6 implies old row 6 minus row 3

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 100 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 300 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 500 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 200 \end{bmatrix}$$

$R_5 \to R_5 - R_4, \qquad R_2 \to R_2 + R_4$

New row 5 implies old row 5 minus row 4, new row 2 implies old row 2 plus row 4 respectively.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 100 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 400 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 300 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 200 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 200 \end{bmatrix}$$

$R_5 \to -(R_5)$means new row 5 implies minus in to old row 5

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 100 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 400 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 300 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 200 \end{bmatrix}$$

$R_5 \to R_6 + R_5, \quad R_3 \to R_3 - R_5$

$R_2 \to R_2 - R_1$

New row 5 implies row 6 plus old row 5, new row 3 implies row 3 minus row 5, new row 2 implies old row 2 minus row 1 respectively.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 100 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 900 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 300 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Evidently, we see that there are an infinite number of solutions. Using the last matrix above, we can write the variables in terms of x_6 and x_7 as

$x_1 = x_6 + 100,$ $x_2 = x_6 - x_7 + 600,$ $x_3 = -x_7 + 900,$

$x_4 = x_6 + 300,$ $x_5 = x_7 - 200$

Since, x_6 must be nonnegative (otherwise) traffic would be moving backwards on a one-way street, we must have $x_6 \geq 300.$

Therefore, to minimize the traffic on New city street vetra street and Newton's street (without backing up traffic), the Kingston City road marshals must allow for a flow of 300vph there and close off traffic on Municipal street between Newton's street and vetra street (because to get x_4 = 300, we must have $x_6 = 0$). Finally, with $x_6 = 0$, we have x_1 =100, $x_2 = -x_7$ + 600, $x_3 = -x_7 + 900,$ x_4 = 300, $x_5 = x_7$ –200.

From the second equation we must have $x_7 \leq$ 600. From the last equation, we must see that $x_7 \geq$ 200. Thus, we obtained the final solution to our problem. To obtain minimum traffic flow at x_4 , we must have $x_1 =$ 100, $0 \leq x_2 \leq$ 400, $300 \leq x_3 \leq$ 700, $x_4 =$ 300, $0 \leq x_5 \leq$ 400, $x_6 = 0$, $200 \leq x_7 \leq 600$

For the rally to be successful without accident and traffic hold ups, the Kingston city road marshals must allow for a flow of 300vph along New city street between Newton's street and Vetra Street. They must allow a flow of 100vph along Newton's street and close off traffic on the Municipal Street between Newton's street and Vetra Street. They must also allow a flow of at most 400vph along vetra street, a flow of 300vph to 700vph along Columnet Street and lastly, a flow of 200vph to 600vph along Municipal street between vetra street and columnet street.

4.0 Conclusion

In this paper, we present the basic definitions of some related terms. We extend the idea of the method of electrical networks analysis to the control of traffic flow in the net of one-way street with the help of the Kirchhoff's current law. We also presented a practical example and solve using row reduction method and finally interpret the result.

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