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Strengthening the LP-Relaxation of Alternative Formulations of Two Stage Capacitated Facility Location Problems

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Abstract

Alternative formulations for the Two Stage Capacitated Facility (Warehouse or Plants) Location Problem (TSCFLP) are compared and shown to be equivalent. The comparison, with main emphasis on strengthening the (linear programming) LP-relaxations is based on some theoretical and computational results. The theoretical aspect compares some relations among the subsets of some constraints of the problem sets. On the other hand, the computational aspect compares the relaxations in terms of the quality of the lower bounds which the original formulations produce when the flow conservation constraints (fcc) are properly represented. Where feasible, these LP based bounds are further strengthened by adding valid inequalities and the problems solved directly on some small and medium size test problems having various characteristics.

Keywords: Mixed integer programming, Facility (Warehouse, or Plant) Location problem, valid inequalities, LP-relaxation.

1.0 Introduction

The facility location problem (FLP) can be classified into different categories depending on the restrictions assumed. In the uncapacitated or simple plant location problem (SPLP), each facility is assumed to have no limits on its capacity. When each facility has a limited capacity the problem is called the capacitated PLP (CPLP). Other sub categories of these problems include the capacitated/uncapacitated PLP with: single source constraints, (CPLPSS), customer's facility preference, P-median problems, aggregate constraints, maximum covering location problems (MCLP) and so on. These are classified as the single stage/level location problems with two decisions to be made. One is the choice of the subset of facilities or plants to open while the second decision is which customers should be assigned to the chosen subset of plants.

When a distribution system consists of facilities on several hierarchically layered levels, where those on higher level can be determined independently of the chosen locations on a lower level, then these type of location problem are called multi stage models, see for example [1].

The TSCFLP is a multi stage model with two stages or levels but practically more than three decision levels. The first or upper most stage is the production plants, where the decision to be made is the choice of which plants to open. The second stage is the distribution depots and the decision to be made in this case is which subset of depots to open. The third stage is the customers and the decision here is to decide which customer should be assigned to which open depots i.e. the open plants to satisfy their demand requirement. Included in this last stage is the decision of the flow of product from the plants to the depots. The problem of choosing the location of facilities in order to serve a set of customers at minimum cost can be encountered in the public sector; (Libraries, health facilities, water treatment plants, the military etc) private sector (factories, telecommunications, Banks, agriculture etc) and managing the environment (waste disposal in chemical industries, tannery, breeding farms etc); Agar and Salhi [2] provided many applications of PLP's across all sectors of societies, while [1] provided various model classifications of PLP and TSCFLP and the methods of solving them.

This paper is concerned with modeling and solving directly the class of mixed integer programming (MIP) problems known as the two stage/level capacitated facility (plants or warehouse) location problem (TSCFLP). The modeling is done by strengthening the linear programming (LP) relaxation of the facility location problem. The model involves choosing the best

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locations for facilities in order to satisfy Customer's demands for certain commodities/products. Given a set of potential locations for facilities and a set of customers, the plant location problem (PLP) is to locate facilities in such a way that the total cost for assigning customers to facilities and satisfying the demand required by customers is minimized. The cost considered is the sum of the fixed costs of opening facilities and the costs for assigning customers to specific facilities.

The paper is organized as follows: section 1.1 gives some notations, definitions and briefly outlines some related works on strengthening the LP relaxation of TSCFLP, section 2 outlines alternative (MIP) formulations for the TSCFLP, section 3 presents the results of the study. Section 4 analyzes the quality of the lower bounds that can be obtained for this problem by LP relaxation and strengthening them. Finally, in section 5, we summarize and conclude our findings.

1.1. Notation, Definitions and some related work

Notation: The following notation is used conventionally except where otherwise stated.

$(S_1, \dots \dots \dots S_k)$ = the set of equality or inequality constraints.

$F(S_1, \dots \dots \dots S_k)$ = the feasible region defined by the constraints$(S_1, \dots \dots \dots S_k)$. $conv(S_1, \dots \dots \dots S_k)$ = the convex hull of the corresponding region.

$v(p)$ = the value for the objective function of problem P.

$I = \{1,\dots,m\}$, the set of plants.

$J = \{1,\dots,n\}$, the set of customers.

$K = \{1,\dots., p\}$, the set of depots;

$c_{kj} =$ Total cost of transportation from depot k to serve customer j, $\forall j \in J, \forall k \in K$

g_k = fixed cost associated with depot k, $\forall k \in K$

$f_i =$ fixed cost associated with plant i $\forall i \in I$

$b_{ik} =$ unit cost of transportation from plant i to depot k, $\forall i \in I, \forall k \in K$

$d_j =$ demand of customer j, $\forall j \in J$

$s_k =$ capacity of depot k, $\forall k \in K$

$a_i =$ capacity of plant i $\forall i \in I$

The decision variables are define as

$x_{kj} =$ fraction of the demand of customer j supplied from depot k, $\forall j \in J, \forall k \in K$

$y_i = \begin{cases} 0 \\ 1 \end{cases}$; 0 if plant i is closed, and 1 if plant i is open, $\forall i \in I$

$w_{ik} =$ units of demand transported from plant i to depot k, $\forall i \in I, \forall k \in K$

$z_k = \begin{cases} 0 \\ 1 \end{cases}$; 0 if depot k is closed, and 1 if depot k is open, $\forall k \in K$

$q_{ikj} =$ Cost of servicing customer j from depot k through plant $i, \forall\ i \in I, j \in J, k \in K$.

$w_{ikj} =$ fraction of the demand of customer j shipped from plant i through depot k.

Definition 1: A linear programming (LP) relaxation is the relaxation of the original (LP) problem formed by removing or dropping the integrality restrictions on the concerned variables. Formally, a relaxation of a minimization problem is defined as follows.

Definition 2: Abdullahi and Sani [3]; Problem $(RZ)\colon \min\{g(x,y)|x,y \in W\}$ is a relaxation of problem $(Z)\colon min\{f(x,y)|x,y \in V\}$, with the same decision variables, iff

i. $F(RZ)$ contains $F(Z)$ i.e $F(RZ) \supseteq F(Z)$
ii. Over $F(Z)$, the objective function of (RZ) dominates (i.e is better than) that of (Z) i.e $\forall x,y \in V, g(x,y) \leq f(x,y)$, where $V \subseteq W$.

iii. It clearly follows that the optimal value of RZ is less than or equal to the optimal value of Z i.e, $\bar{z}(.) \leq z(.)$, (in case ***Table 3: Sensitivity analysis*** of minimization) since RZ has more feasible solutions than Z; where $z(.)$ is the optimal integer objective value, while $\bar{z}(.)$ is the optimal objective value of the relaxed problem.

Definition 3: Magnanti and Wong [4]: A polyhedron P$\subseteq \mathbb{R}_+^n \times \mathbb{Z}_+^m$; where $n = (i \times k) \times (k \times j), m = (i \times k)$ is a formulation for a set X^{TSCFL}, if $X^{TSCFL} = P \cap (\mathbb{R}_+^n \times \mathbb{Z}_+^m)$. This definition indicates the existence of many formulations for a set X. and this raises the questions about "good" and 'not so good" formulations.

Definition 4: Magnanti and Wong [4]: Given a set $X \subseteq \mathbb{R}^n$ and two formulations P_1 and P_2 for X, we say that P_1 is better than P_2, if $P_1 \subseteq P_2$. A formulation P is called ideal if $P \subseteq conv(X)$.

Definition 5. Magnanti and Wong [4]: Problem P is said to dominate (or is a stronger formulation than) problem Q if $v(P) \geq v(Q)$ for all $\in Y$, with a strict inequality for at least one point $y \in Y$ ..

Definition 6. Magnanti and Wong [4]: Problem formulations P and Q are said to be equivalent MIP representation of the same problem if $v(P)(y,z) = v(Q)(y,z) for\ all\ y_{i\in I}, z_{k\in K}$. i.e. The two models have the same integer variables and may have different continuous variables and constraints , but always give the same objective function values for any feasible assignment of the integer variables.

We discuss the LP relaxations of various alternative formulations of the TSCFLP strengthened by: (i). Representing the flow conservation constraints (*fcc)* correctly [5]. (ii).The valid inequalities based on Davis and Ray (*D&R*) [3]. (iii).The valid inequalities based on Ro and Tha (*R&T*) [6]. Analogous work on strengthening the LP relaxation of the TSCFLP based on Knapsack cover, flow cover and fixed charge path inequalities are presented in [7].

The main purpose of LP relaxation in solving MIP is to provide an optimal value which in turn provides a lower bound (in case of minimization) on the optimal value of the corresponding MIP. While other relaxations such as those based on lagrangian duality, semi definite programming and decomposition techniques are undoubtedly useful in special situations. LP relaxation generally gives reasonably tight bounds, and the methodology for solving LPs is very efficient and reliable [8]. Works on strengthening the LP relaxations of CFLP and hence the TSCFLP are reported in the literature. Most of the reports are centered on CFLP, but as usual the TSCFLP is an extension of CFLP and most of the results found for CFLP are valid for TSCFLP [7 and 9].

Studies on LP relaxations of alternative formulations of TSCFLP where the same feasible set is represented by different sets of constraints (which may provide different lower bounds) in the context of LP relaxation is not reported in the literature as far as we know. In [10], for example, work on single-client CFLP was considered and they gave extended flow cover inequalities with uniform capacities and their algorithm is an approximation with integrality gap = 1. In [5], adapted flow cover inequalities for a general CFLP was considered. They used this cutting plane to tighten the formulation, thereby providing a better lower bound with integrality gap of less than one percent. Theirs is not an approximation algorithm like the former case. In [11], an approximation algorithm based on covering inequalities for a single client CFLP with integrality gap ≤ 2 was considered. In our case, the flow conservation constraint (*fcc*) [5], with the valid inequalities of Davis and Ray, (*D&R)* [3], are appended to various alternative flow formulations of the TSCFLP, while the valid inequalities due to Ro and Tcha (*R&T*) [6], were incorporated to the multicommodity formulation of TSCFLP. The effect of all these is discussed in sections 3 and 4.

On the comparison of alternative formulations of TSCFLP; in [12], an optimization problem over the set of lagrangian relaxations of two alternative formulations of TSCFLP with the objective of finding the relaxation that produces the best dual bound was considered. Also in [6], two alternative mathematical model formulations for the two-level distribution and waste disposal problem with capacity constraints (which is a special case of TSCFLP) are analyzed. They have shown that both formulations are equivalent. Also, comparison of alternative formulation can be seen in [13] but, in the context of comparing several lagrangian relaxations of the formulations of two-stage uncapacitated facility location problem. In the seminal work presented in [4], two general alternative formulations of MIP problem were considered but, with the objective of theoretically outlining model formulation selection criterion in the context of accelerating Benders's Decomposition. In this paper we considered seven flow formulations of alternative mathematical models, and one multicommodity formulation of TSCFLP, with the objective of analyzing their LP relaxations.

2. Model Formulations

Modeling the two Level/stage problems is slightly less straightforward than the one stage problem. There are two obvious ways of formulating the problem: "flow formulation" and the "multi commodity formulation". In the flow formulation we consider the flow at each level, and require conservation of flow between levels. It can be proved [5], that the LP relaxation of the multicommodity formulation is at least as strong as the LP relaxation of the flow formulation. A draw back with the multicommodity formulation is, however, that it grows rapidly as the size of the problem instance grows, see table 2 for example (cf. Table 1).

FLOW FORMULATION: TSCFLP can be stated as follows: A single product is produced at some facilities, plants or warehouses in order to satisfy customer demands. The product is transported from these plants (or major plants) to some depots (or minor plants) and then to the customers. The capacities of plants and depots are limited. The problem formulation for TSCFLP, as presented in [14] can be stated as follows:

$$Z_{ff1} = \min\left\{\sum_{i\in I}\sum_{k\in K} b_{ik}\,w_{ik} + \sum_{k\in K}\sum_{j\in J} c_{kj}\,x_{kj} + \sum_{i\in I} f_i\,y_i + \sum_{k\in K} g_k\,z_k : (w,x,y,z)\in X^{TSCFL}\right\}, \quad (1\text{-}1)$$

where

$$X^{TSCFL} = \{\ (w,x,y,z) \in \mathbb{R}_+^{i\times k} \times \mathbb{R}_+^{k\times j} \times \mathbb{Z}_+^{i} \times \mathbb{Z}_+^{k}:$$

$$\sum_{k\in K} x_{kj} = 1 \qquad \forall j\in J \qquad (1\text{-}2)$$

$$\sum_{j\in J} d_j\,x_{kj} \le s_k\,z_k \qquad \forall k\in K \qquad (1\text{-}3)$$

$$x_{kj} - z_k \le 0 \qquad \forall j\in J, \forall k\in K \qquad (1\text{-}4)$$

$$\sum_{k\in K} w_{ik} \le a_i\,y_i \quad \forall i\in I \qquad (1\text{-}5)$$

$$\sum_{i\in I} w_{ik} = \sum_{j\in J} d_j\,x_{kj} \qquad \forall k\in K \qquad (1\text{-}6)$$

$$w_{ik} \ge 0 \qquad \forall i\in I, \forall k\in K \qquad (1\text{-}7)$$

$$y_i, z_k \text{ integer} \qquad \forall i\in I, \forall k\in K \qquad (1\text{-}8)$$

$$0\le x_{kj}\le 1, \quad 0\le y_i\le 1, \quad 0\le z_k\le 1, \quad \forall i,k,j\ \} \qquad (1\text{-}9)$$

The objective function (1-1) minimizes the sums of the fixed costs of opening both plants and depots, and the transportation costs of shipping demand from plants to depots and from depots to customers. The constraint (1-2) ensures that each customer's demand is fully met by the depots. Constraint (1-3) ensures that open depots do not supply more than their capacity, i.e, for each depot the sum of the demand of the customers it is supplying is less than or equal to its capacity. Constraint (1-4) ensures that customers are only served from open depots. Constraint (1-5) guarantees that open plants do not supply more than their capacities, i.e, for each plant the sum of the demand leaving it is less than or equal to the capacity it can hold. Constraint (1-6) indicates conservation of flow constraints for the depots. That is, for each depot the amount of demand entering the depot from the plants is equal to the demand leaving the depot to be transported to the customers. Constraint (1-7) consists of non negativity constraints on the amount of demand transported from plants to depots. Constraint (1-8) consists of integrality constraints on both plants and depots. Constraints (1-9) are non negativity and simple upper bound constraints restricting the fractional values of customers demand.

Surrogate constraints (1-10) and (1-11) can be added as follows:

$$\sum_{i\in I}\sum_{k\in K} w_{ik} \le \sum_{i\in I} a_i\,y_i \qquad (1\text{-}10)$$

$$\sum_{k\in K} s_k\,z_k \ge \sum_{j\in J} d_j \qquad (1\text{-}11)$$

(1-10) is derived by summing (1-5) over all i plants and states that the total capacity of the plants is at least as large as the total demand being transported from them to the depots. (1-11) is derived by summing (1-3) over all k depots and using the equalities (1-2) and ensures that the total capacities of the depots is at least as large as the total demand being transported from them to the customers. These two constraints are redundant in the original formulation but strengthen some of the relaxations. The second formulation of the TSCFLP with the surrogate constraints added is:

$$Z_{ff2} = \min\left\{\sum_{i\in I}\sum_{k\in K} b_{ik}\,w_{ik} + \sum_{k\in K}\sum_{j\in J} c_{kj}\,x_{kj} + \sum_{i\in I} f_i\,y_i + \sum_{k\in K} g_k\,z_k : (w,x,y,z)\in X^{TSCFL}\right\} \qquad (2\text{-}1)$$

where

$$X^{TSCFL} = \{\ (w,x,y,z) \in \mathbb{R}_+^{i\times k} \times \mathbb{R}_+^{k\times j} \times \mathbb{Z}_+^{i} \times \mathbb{Z}_+^{k}: (1-2)\ to\ (1-9)\ plus$$

$(1-10), (1-11)\}$. Adding these surrogate constraints will not change/or better the optimal objective values of both the MIP and its relaxation, rendering them redundant in this case, but they are useful in strengthening other relaxations like lagrangean relaxation [3].

In the work of [12], two formulations of TSCFLP were presented, the first one contained the valid inequalities of Davis and Ray [15]; and is given as follows:

$$Z_{ff3} = \min\left\{\sum_{i\in I}\sum_{k\in K} b_{ik} w_{ik} + \sum_{k\in K}\sum_{j\in J} c_{kj} x_{kj} + \sum_{i\in I} f_i y_i + \sum_{k\in K} g_k z_k : (w,x,y,z) \in X^{TSCFL}\right\}, \quad (3\text{-}1)$$

where

$$X^{TSCFL} = \{ (w,x,y,z) \in \mathbb{R}_+^{i\times k} \times \mathbb{R}_+^{k\times j} \times \mathbb{Z}_+^{i} \times \mathbb{Z}_+^{k}:$$

$$\sum_{k\in K} w_{ik} \le a_i \qquad \forall i \in I \qquad (3\text{-}2)$$

$$\sum_{i\in I} w_{ik} \le s_k \qquad \forall k \in K \qquad (3\text{-}3)$$

$$\sum_{k\in K} x_{kj} \ge d_j \qquad \forall j \in J \qquad (3\text{-}4)$$

$$\sum_{i\in I} w_{ik} \ge \sum_{j\in J} x_{kj} \qquad \forall k \in K \qquad (3\text{-}5)$$

$$w_{ik} \le m_{ik} y_i \qquad \forall i \in I, \forall k \in K, \qquad (3\text{-}6)$$

$$x_{kj} \le l_{kj} z_k \qquad \forall k \in K, \forall j \in J \qquad (3\text{-}7)$$

$$w_{ik}, x_{kj} \ge 0 \qquad \forall i \in I, \forall j \in J, \forall k \in K \qquad (3\text{-}8)$$

$$y_i, z_k \in \{0, \quad 1\} \qquad \forall i \in I, \forall k \in K\} \qquad (3\text{-}9)$$

Where, $m_{ik} = \min\{a_i, s_k\}$ *and* $l_{kj} = \min\{s_k, d_j\}$ $\forall i \in I, \forall j \in J, \forall k \in K$ are upper bounds for the respective flows [12]. Valid inequalities (3-6) and (3-7) are based on [15]. Thus constraints (3-6,3-7) ensures that total flow between plant i and customer j can never exceed the minimum of customer $j's$ demand and the capacity at plant i, and total product flow between plant i and depot k can never be larger than the minimum of the capacity at depot k, and the maximum production generated at plant i respectively. Constraints (3-4) and (3-5) are fulfilled as equalities for an optimal solution of (3-1)-(3-9). The second formulation of TSCFLP presented in [12] arises when the valid inequalities of [15], (constraints (3-6) and (3-7)) are written in a more concise form yielding an equivalent formulation of the TSCFLP, given as follows:

$$Z_{ff4} = \min\left\{\sum_{i\in I}\sum_{k\in K} b_{ik} w_{ik} + \sum_{k\in K}\sum_{j\in J} c_{kj} x_{kj} + \sum_{i\in I} f_i y_i + \sum_{k\in K} g_k z_k : (w,x,y,z) \in X^{TSCFL}\right\}, \quad (4\text{-}1)$$

where

$$X^{TSCFL} = \{ (w,x,y,z) \in \mathbb{R}_+^{i\times k} \times \mathbb{R}_+^{k\times j} \times \mathbb{Z}_+^{i} \times \mathbb{Z}_+^{k}:$$

$$\sum_{k\in K} x_{kj} \ge d_j \qquad \forall j \in J \qquad (4\text{-}2)$$

$$\sum_{i\in I} w_{ik} \ge \sum_{j\in J} x_{kj} \qquad \forall k \in K \qquad (4\text{-}3)$$

$$\sum_{k\in K} w_{ik} \le a_i y_i \qquad \forall i \in I \qquad (4\text{-}4)$$

$$\sum_{i\in I} w_{ik} \le s_k z_k \qquad \forall k \in K \qquad (4\text{-}5)$$

$$w_{ik}, x_{kj} \ge 0 \qquad \forall i \in I, \forall j \in J, \forall k \in K \qquad (4\text{-}6)$$

$$y_i, z_k \in \{0, \quad 1\} \qquad \forall i \in I, \forall k \in K\} \qquad (4\text{-}7)$$

A tight formulation of the TSCFLP was given in [5]. This formulation differs from the previous formulations on the way the flow conservation constraint (5-4) below is presented, which differs from constraints (1-6), (3-5) and (4-3). Mathematically the flow conservation constraints { (1-6),(3-5), (4-3)} and (5-4) are equivalent, but computationally (5-4) as it is written, tightens the formulation and yields a better lower bound than those formulations (Z_{ff1}, Z_{ff2}, Z_{ff3} and Z_{ff4}) above. The software CPLEX, XPRESS and MINTO, [5], recognizes the path structure when the constraint is presented in the correct form. The Aardal formulation is as follows:

$$Z_{ff5} = \min\left\{\sum_{i\in I}\sum_{k\in K} b_{ik}\, w_{ik} + \sum_{k\in K}\sum_{j\in J} c_{kj}\, x_{kj} + \sum_{i\in I} f_i\, y_i + \sum_{k\in K} g_k\, z_k : (w,x,y,z)\in X^{TSCFL}\right\}, \qquad (5\text{-}1)$$

where

$$X^{TSCFL} = \{ (w,x,y,z) \in \mathbb{R}_+^{i\times k} \times \mathbb{R}_+^{k\times j} \times \mathbb{Z}_+^{i} \times \mathbb{Z}_+^{k}:$$

$$\sum_{k\in K} w_{ik} \le a_i y_i \qquad \forall i\in I \qquad (5\text{-}2)$$

$$\sum_{i\in I} w_{ik} \le s_k z_k \qquad \forall k\in K \qquad (5\text{-}3)$$

$$\sum_{i\in I} w_{ik} - \sum_{j\in J} x_{kj} = 0 \qquad \forall k\in K \qquad (5\text{-}4)$$

$$\sum_{k\in K} x_{kj} = 1 \qquad \forall j\in J \qquad (5\text{-}5)$$

$$\sum_{j\in J} x_{kj} \le s_k z_k \qquad \forall k\in K \qquad (5\text{-}6)$$

$$x_{kj} \le d_j z_k \qquad \forall k\in K \qquad (5\text{-}7)$$

$$y_i \le 1 \quad \forall i\in I,\; z_k \le 1 \quad \forall k\in K \qquad (5\text{-}8)$$

$$w_{ik}, x_{kj} \ge 0 \qquad \forall i\in I, \forall j\in J, \forall k\in K\} \qquad (5\text{-}9)$$

For instance, it is important to write constraints (5-4) as $\sum_{i\in I} w_{ik} - \sum_{j\in J} x_{kj} = 0$, and not as $\sum_{j\in J} x_{kj} - \sum_{i\in I} w_{ik} = 0$, since the sign of a variable indicates whether or not it represents inflow or outflow.

Another alternative formulation was presented in [1] and it is as follows:

$$Z_{ff6} = \min\left\{\sum_{i\in I}\sum_{k\in K} b_{ik}\, w_{ik} + \sum_{k\in K}\sum_{j\in J} c_{kj}\, x_{kj} + \sum_{i\in I} f_i\, y_i + \sum_{k\in K} g_k\, z_k : (w,x,y,z)\in X^{TSCFL}\right\}, \qquad (6\text{-}1)$$

where

$$X^{TSCFL} = \{ (w,x,y,z) \in \mathbb{R}_+^{i\times k} \times \mathbb{R}_+^{k\times j} \times \mathbb{Z}_+^{i} \times \mathbb{Z}_+^{k}:$$

$$\sum_{k\in K} x_{kj} = 1 \qquad \forall j\in J \qquad (6\text{-}2)$$

$$\sum_{j\in J} d_j x_{kj} \le s_k z_k \qquad \forall k\in K \qquad (6\text{-}3)$$

$$x_{kj} - z_k \le 0 \qquad \forall j\in J, \forall k\in K \qquad (6\text{-}4)$$

$$\sum_{k\in K} w_{ik} \le a_i y_i \qquad \forall i\in I \qquad (6\text{-}5)$$

$$\sum_{i\in I} w_{ik} = \sum_{j\in J} d_j\, x_{kj} \qquad \forall k\in K \qquad (6\text{-}6)$$

$$w_{ik} - \min\{a_i, s_k\} y_i \le 0 \qquad \forall i\in I, \forall k\in K \qquad (6\text{-}7)$$

$$\sum_{k\in K} s_k z_k \ge \sum_{j\in J} d_j \qquad \forall k\in K, \forall j\in J \qquad (6\text{-}8)$$

$$\sum_{i\in I} a_i y_i \ge \sum_{j\in J} d_j \qquad \forall i\in I, \forall j\in J \qquad (6\text{-}9)$$

$$w_{ik}, x_{kj} \ge 0 \qquad \forall i\in I, \forall j\in J, \forall k\in K \qquad (6\text{-}10)$$

$$y_i, z_k \in \{0, \quad 1\} \qquad \forall i\in I, \forall k\in K\} \qquad (6\text{-}11)$$

We contemplate the following equivalent formulation by dropping some unbinding constraints, and restructuring some which are redundant in the context of LP relaxation:

$$Z_{ff7} = \min\left\{\sum_{i\in I}\sum_{k\in K} b_{ik}\, w_{ik} + \sum_{k\in K}\sum_{j\in J} c_{kj}\, x_{kj} + \sum_{i\in I} f_i\, y_i + \sum_{k\in K} g_k\, z_k : (w,x,y,z)\in X^{TSCFL}\right\}, \quad (7\text{-}1)$$

where

$$X^{TSCFL} = \{ \ (w,x,y,z) \in \mathbb{R}_+^{i\times k} \times \mathbb{R}_+^{k\times j} \times \mathbb{Z}_+^{i} \times \mathbb{Z}_+^{k}:$$

$$\sum_{k\in K} x_{kj} = 1 \qquad \forall j\in J \qquad (7\text{-}2)$$

$$\sum_{j\in J} x_{kj} \le s_k\, z_k \qquad \forall k\in K \qquad (7\text{-}3)$$

$$\sum_{i\in I} w_{ik} = \sum_{j\in J} d_j\, x_{kj} \qquad \forall k\in K \qquad (7\text{-}4)$$

$$\sum_{k\in K} w_{ik} \le a_i\, y_i \qquad \forall i\in I \qquad (7\text{-}5)$$

$$w_{ik}, x_{kj} \ge 0 \qquad \forall i\in I, \forall j\in J, \forall k\in K \qquad (7\text{-}6)$$

$$y_i, z_k \in \{0,1\}, \qquad \forall i\in I, \forall j\in J, \forall k\in K \ \} \qquad (7\text{-}7)$$

MULTI COMMODITY FORMULATION: In the multi commodity formulation we consider the flow on the path (i,k,j) where, i, is a major plant , k is a minor (depot) plant and j is the customer, we let w_{ikj} denote the fraction of demand d_j being routed via path $i \to k \to j$, and defines the cost at demand point $j \in J \; via \; path \; i \to k \to j$ as $q_{ikj} = b_{ikj} d_j + c_{kj}$. This formulation allows us to model situations where cost depends on both the major plant, i, through minor plant k (or depot) and demand point j. Such cases occur in practice if for instance flow rates from source i to depot k are less than the sum of flow rates from $i \; to \; k \; plus \; k \; to \; j$. Apparently, flow formulation (*ff*), has more advantages to multi commodity formulation (*mcf*) if the cost q_{ikj} can be split in to two parts $b_{ikj} \; and \; c_{kj}$ because *ff* has far fewer decision variables (cf. Table 2), while the LP relaxation of both models are equivalent.

Klose and Drexl [1] and Aardal [5] proved that the LP relaxation of the multi commodity formulation is at least as strong as the LP relaxation of the flow formulation, and for many instances the difference can be quite large. An equivalent formulation of the TSCFLP based on path variables w_{ikj} can be given as follows:

$$Z_{mcf} = \min \quad \sum_{i\in I}\sum_{k\in K}\sum_{j\in J} q_{ikj}\, w_{ikj} + \sum_{i\in I} f_i\, y_i + \sum_{k\in K} g_k\, z_k : (w,x,y,z)\in X^{TSCFL} \qquad (8-1)$$

where

$$X^{TSCFL} = \{ \ (w,x,y,z) \in \mathbb{R}_+^{i\times k\times j} \times \mathbb{Z}_+^{i} \times \mathbb{Z}_+^{k}:$$

$$\sum_{i\in I}\sum_{k\in K} w_{ikj} = 1 \qquad \forall j\in J \qquad (8-2)$$

$$\sum_{k\in K}\sum_{j\in J} d_j\, w_{ikj} \le a_i\, y_i \qquad \forall i\in I \qquad (8-3)$$

$$\sum_{i\in I}\sum_{j\in J} d_j\, w_{ikj} \le s_k\, z_k \qquad \forall k\in K \qquad (8\text{-}4)$$

$$\sum_{k\in K} w_{ikj} \le y_i \qquad \forall i\in I \ \ \forall j\in J \qquad (8\text{-}5)$$

$$\sum_{i\in I} w_{ikj} \le z_k \qquad \forall k\in K \qquad \forall j\in J \qquad (8\text{-}6)$$

$$\sum_{k\in K} s_k z_k \geq \sum_{j\in J} d_j \qquad \forall k \in K \qquad \forall j \in J \qquad (8\text{-}7)$$

$$\sum_{i\in I} a_i y_i \geq \sum_{j\in J} d_j \qquad \forall i \in I \quad \forall j \in J \qquad (8\text{-}8)$$

$$w_{ikj} \geq 0 \qquad \forall i \in I \ \ \forall k \in K \qquad \forall j \in J \qquad (8-9)$$

$$y_i \in \{0,1\} \qquad \forall i \in I \qquad (8-10)$$

$$z_k \in \{0,1\} \qquad \forall k \in K\} \qquad (8-11)$$

For more on this see [1] and [6]. The objective function (8-1) gives the total cost consisting of the cost of assigning customers to facilities plus the cost of establishing facilities and cost of opening depots. Constraints (8-2) ensure that demand is satisfied completely. Constraints (8-3), (8-4) take care of scarce capacities of facilities on both levels. Aggregate capacity constraints (8-7) and (8-8) are redundant but probably useful in order to tighten some relaxations such as lagrangean relaxation other than LP relaxation. Constraints (8-5) and (8-6) are valid inequalities that also strengthen LP relaxation.

3. Computational Results

The problem instances are from [14]. In Table 1 and Table 2 we present the problem characteristics for both the instances used for the flow and multi commodity formulations of TSCFLP respectively. We considered small and medium size problems of the type, $a \times b \times c$, where $a, b\ and\ c$ denotes the number of major Facilities (Plants), number of minor Facilities (Depots) and number of Clients (Customers) respectively.

Table 1 problem characteristics for the flow formulation

Problem	Size	Number of variables	Number of Constraints	Number of nonzeros
A	3x5x10	73	73	288
B	5x8x25	253	246	1093
C	5x10x25	315	300	1365
D	5x16x25	501	462	2181

Table 2 problem characteristics for the multi commodity formulation

Problem	Size	Number of variables	Number of Constraints	Number of nonzeros
MC1	3x5x10	158	100	847
MC2	5x8x25	1013	365	5351
MC3	5x10x25	1265	417	6655
MC4	5x16x25	2021	573	10567

In Table 2, the same group of instances as in Table 1 is formulated using variables w_{ikj} denoting the flow on the path (i, k, j), and variables w_{ik}, x_{kj}, y_i, z_k as in the flow formulation, for the multi commodity formulation. In Table 3, we show the results from solving the instances by pure Branch and Bound only, and in Table 4, we show the results from solving the instances by representing the flow conservation constraint (fcc) correctly. We use the notation abbccd for the instances, where a denotes the number of plants, bb the number of depots, cc the number of customers and d the number of the instances in the set having the same size, we also use $z(.)$ and $\bar{z}(.)$ denoting the optimum integer and continuous (LP relaxed) solutions of the problems respectively. $\xi_{bound} = \frac{z(.)-\bar{z}(.)}{z(.)} * 100\%$ is the measure of the relative quality of the bounds, often referred in the literature as duality gap (%) (or % duality gap), [16, 17]. For the computation we coded the formulations according to the syntax of AMPL [18] and use IBM-CPLEX 12.5.0 solver, implemented on a HP corei3 processor 2.27 GHz 4 GB RAM PC.

The following implication of definitions (5) and (6) above stands.

Theorem 1. Magnanti and Wong [4] Suppose P and Q are equivalents formulation of a MIP. P dominates (is superior to) Q iff $v(P) \geq v(Q)\ \ for\ all\ \ y_i, z_k \in conv(y \times z)$ with a strict inequality for at least one $y_i, z_k \in conv(y \times z)$.

Proof: $\Leftarrow$ Suppose $(P) \geq v(Q)\ , for\ all\ \ y_i, z_k \in conv(y \times z)$, then Q does not have any valid inequality w.r.t. P. but there exists at least $y_0, z_0 \in conv(y, z)$ such that $v(P)(y_0, z_0) > v(Q)(y_0, z_0)$ implies that P has a valid inequality that is not equal to any valid inequality in P.

$\Rightarrow$ Now if P dominates (is superior) Q, then Q, by definition of dominance does not have any valid inequality that is equal to any inequality w.r.t. Q, this implies that $v(P) \geq v(Q)\ \ for\ all\ \ y_i, z_k \in conv(y \times z)$, and there exist a $y_0, z_0 \in conv(y, z)$ such that $v(P)(y_0, z_0) > v(Q)(y_0, z_0)$.

The computational results are given in Table(s) 3 to Table(s) 5. For example in Table 3b and 3c, Z_{ff3} and Z_{ff4} have the same optimal integer objective values for the flow formulation, in the same vain $Z_{ff`1}$ and Z_{ff6} have the same optimal integer objective values. Similarly in Table 4b , 4c and 4e, Z_{ff3}, Z_{ff4} and Z_{ff7} have the same optimal integer objective values respectively. In order to highlight further the relationship that exists between the various formulations, Figures 1 to 6 are graphs of some selected parameters of the computations. For example, Figures 1 and 2 are graphs of MIP and LP CPU time for the problems size for all the flow formulations respectively. From these graphs we can conclude that Z_{ff1} has higher CPU time for both MIP and LP-relaxations. Among the formulations, for example, Figure 5 and Figure 6 shows that Z_{ff3} & Z_{ff4} have higher optimal integer objective function costs, followed by Z_{ff1} & Z_{ff6}, wile, Z_{ff7} and Z_{ff5} have the least costs respectively. Also Z_{ff1} has higher number of MIP and dual simplex iterations than Z_{ff6}. Analogous interpretations can be given to the rest of the Tables and Figures.

4. Results Analysis

Here we compare the relative quality of the optimal integer values of the formulations $Z_{ff}\ and\ Z_{mcf}$ and their LP-relaxations. For this comparison we introduce the following additional notations. When formulation $Z_{ff}\ and\ Z_{mcf}$ were solve by B&B only, the values for the optimal integer objective and LP-relaxations will be denoted by $z_{ff}^{A}(.)$, $z_{mcf}^{A}(.)$ and $\bar{z}_{ff}^{A}(.), \bar{z}_{mcf}^{A}(.)$ respectively. And when they are solved by B&B with correct representation of the flow conservation constraint, the values for the optimal integer objective and LP-relaxations will be denoted by $z_{ff}^{B}(.), z_{mcf}^{B}(.)$ and $\bar{z}_{ff}^{B}(.), \bar{z}_{mcf}^{B}(.)$ respectively. From Table(s) 3 to Table 5 we have the following results:

Lemma 1 $Z_{ff} = Z_{mcf}$, that is, both the flow and the multi commodity formulation of the TSCFLP are equivalent.

Proof: The following equivalence relations hold by definition [6]:

(a) $x_{kj} = \sum_{k\in K} d_j w_{ikj}$

(b) $w_{ik} = \sum_{j\in J} d_j w_{ikj}$

(c) $\sum_{k\in K} x_{kj} = d_j = 1$

(d) $c_{kj} + b_{ik} d_j = q_{ikj}$

Given these equivalence relations we prove that $z_{ff1}(.)$ is equivalent to $z_{mcf}(.)$.

i. The objectives (1-1) and (8-1) in both formulations are equivalent:

$$\sum_{i\in I}\sum_{k\in K}\sum_{j\in J} q_{ikj} w_{ikj} + \sum_{i\in I} f_i y_i + \sum_{k\in K} g_k z_k$$ then from (d) we have:

$$= \sum_{i\in I}\sum_{k\in K}\sum_{j\in J} (b_{ikj} d_j + c_{kj}) d_j w_{ikj} + \sum_{i\in I} f_i y_i + \sum_{k\in K} g_k z_k$$ then from (a) and (b) we have

$$= \sum_{i\in I}\sum_{j\in J} c_{kj} \sum_{k\in K} d_j w_{ikj} + \sum_{i\in I}\sum_{k\in K} b_{ik} \sum_{j\in J} d_j w_{ikj} + \sum_{i\in I} f_i y_i + \sum_{k\in K} g_k z_k$$

$$= \sum_{i\in I}\sum_{k\in K} b_{ik} w_{ik} + \sum_{i\in I}\sum_{j\in J} c_{kj} x_{kj} + \sum_{i\in I} f_i y_i + \sum_{k\in K} g_k z_k$$

Demand constraints (1-2) and (8-2) are equivalent. Since for each j = 1, …, n

$$\sum_{k\in K} x_{kj} = \sum_{i\in I}\sum_{k\in K} d_j w_{ikj} \quad \forall j \in J$$ then from (c),

$$d_j = \sum_{i\in I}\sum_{k\in K} d_j w_{ikj} \quad \forall j \in J \Rightarrow \sum_{i\in I}\sum_{k\in K} w_{ikj} = 1$$

ii. Flow conservation constraints (1-6) is satisfied in *MCF* $\sum_{i\in I} w_{ik} = \sum_{j\in J} d_j\, x_{kj} \quad \forall k \in K$ for each k = 1,…, p

$\sum_{j\in J} d_j\, x_{kj}$ is the demand for customer $j\ from\ depot\ k$, and by definition, this must come from plant i. From (b) above,

$$w_{ik} = \sum_{j\in J} d_j\, w_{ikj}$$, taking the sum over k, we have $\sum_{k\in K} w_{ik} = \sum_{k\in K} \sum_{j\in J} d_j\, w_{ikj}$ then from (a)

and (8-3) $\Rightarrow \quad \le a_i y_i \quad \forall i \in I$

iii. Plant capacity constraints (1-5) and (8-3) are equivalent.

$\sum_{k\in K} w_{ik} \le a_i y_i \ \forall i \in I \quad \sum_{k\in K}\sum_{j\in J} d_j\, w_{ikj} \le a_i y_i \ \forall i \in I$, the right hand sides of the two inequalities are equivalent, we show that the left hand sides are also equivalent. From (b)

$$\sum_{k\in K} \left[\sum_{j\in J} d_j\, w_{ikj}\right] = \sum_{k\in K} w_{ik} \quad \forall i \in I \Rightarrow \sum_{k\in K} w_{ik} \le a_i y_i \quad \forall i \in I$$

Depot capacity constraint (1-3) and (8-4) are equivalent.

$$\sum_{j\in J} d_j\, x_{kj} \le s_k z_k \quad \forall k \in K, \quad \sum_{i\in I}\sum_{j\in J} d_j\, w_{ikj} \le s_k z_k \ ,\forall k \in K$$

Similarly the right hand sides of the two inequalities are equivalent. We show the left hand sides also are equivalent. From (1-6) $\sum_{j\in J} d_j\, x_{kj} = \sum_{i\in I} [w_{ik}] \quad \forall k \in K$ from (b) $= \sum_{i\in I}\sum_{j\in J} d_j\, w_{ikj}$

iv. Constraints (8-7) ,(8-8) are redundant just like (1-10), (1-11). The rest: that is (1-7, 1-8, &1-9) and (8-9, 8-10, and 8-11) are the non-negativity and integrality restrictions on the respective variables of the two formulations respectively.

Lemma 2 let $Z_{ff1}, \dots \dots \dots, Z_{ff7}\ and\ Z_{mcf}$ be formulations for the set $X^{TSCFL} \subseteq \mathbb{R}_+^n \times \mathbb{Z}_+^m$; where some $Z_{ffi-p}; 1 \le p < i\ ; i = 1, \dots \dots \dots, 7\ and\ Z_{mcf}$ are better than some Z_{ff}. If we consider the MIP problem $Z_{ff} = Z_{mcf} = min\{C^T x : x \in X^{TSCFL}\}$ where $x = (w, x, y, z)$ and denote by $z_{ff}(.) = min\{C^T x : x \in Z_{ff}\}$, $\bar{z}_{ff}(.) = min\{C^T x : x \in Z_{ff},\ X^{TSCFL} \subseteq \mathbb{R}_+^n \times \mathbb{R}_+^m\}$, the values of the associated optimal integer and LP-relaxations of problem Z_{ff} or Z_{mcf}, then the following hold:

i. $\bar{z}_{ff7}^A(.) < \bar{z}_{ff1}^A(.) \le \bar{z}_{mcf}^A(.) \le \bar{z}_{ff6}^A(.) \le \bar{z}_{ff4}^A(.) \le \bar{z}_{ff3}^A(.)$
ii. $z_{ff1}^A(.) = z_{ff6}^A(.) = z_{mcf}^A(.)$
iii. $z_{ff3}^A(.) = z_{ff4}^A(.)$
iv. $z_{ff7}^A(.) \le \{(ii), (iii)\}$ dominates $z_{ff1}^A(.),\ z_{ff6}^A(.),\ z_{mcf}^A, z_{ff3}^A(.)\ and\ z_{ff4}^A(.)$
v. $\bar{z}_{ff4}^B(.) = \bar{z}_{ff5}^B(.) = \bar{z}_{ff7}^B(.)$
vi. $\bar{z}_{ff1}^B(.) \le \bar{z}_{ff6}^B(.)$
vii. $z_{ff4}^B(.) = z_{ff5}^B(.) = z_{ff7}^B(.)$
viii. $z_{ff1}^B(.) \le z_{ff6}^B(.)$

Proof: i. constraints (3-5) and (4-3) is a relaxation of constraint (1-6), by definition 1 the first inequality holds; by lemma 1 the second inequality holds; Z_{ff3} implied Z_{ff4} and by definition 5 and the results on Table 3b and 3c, the relationships hold.

ii. Z_{ff1} and Z_{ff6} are both flow formulation, by lemma 1, both are equivalent.

iii. from Table 3b and 3c, and ii above, the relation hold.

iv. This follows from definition 4, 5 and theorem 1. And Table 3f shows the values of the optimal objective function for the instances solved are at least better than Z_{ff1}, Z_{ff6}, Z_{ff3} and Z_{ff4}.

v. from lemma 1, (c) constraint (4-2) is equivalent to constraints (5-5) and (7-2), which say all customers' demands must be fully met, hence the result.

vi. This follows from definition 5 and the results for solving the instances in Table 4a and 4d respectively.

vii. This follows from a direct consequence of (v) above.

viii. (vi) implies (viii), as in (vii) above.

Table 3a: Results from solving the instances by Branch and Bound only for Z_{ff1}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	55152.8619	49815.26011	9.7	0	62	51	0.0938	0.0306
305102	67061.31787	□4622.98088	3.6	0	52	57	0.1248	0.0466
305103	36575.5455	33912.30113	7.3	0	51	45	0.1402	0.031
508251	63392.8448	59633.96184	5.9	0	197	92	0.234	0.0626
508252	53793.94231	51297.94476	4.6	0	119	85	0.296	0.1092
508253	22503.2	20344.14025	1.0	0	141	130	0.3736	0.093
510251	62129.54196	58319.22228	6.1	0	207	142	0.4672	0.109
510252	43234.99827	39112.08207	9.5	0	157	128	0.4518	0.1556
510253	109651.0481	106048.1401	3.3	0	181	130	0.654	0.1716
516251	135244.858	132656.3339	1.9	0	218	198	0.6074	0.2182
516252	77216.70671	70387.46144	8.8	1	264	143	0.888	0.2492
516253	28860.38889	23478.0224	1.9	0	310	170	1.0286	0.234

Average duality gap 7.4%; Note that Z_{ff2} has the same optimal solution with Z_{ff1}

Table 3b: Results from solving the instances by Branch and Bound only for Z_{ff3}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	64116	63938.95607	0.28	0	53	46	0.0622	0
305102	83841	82835.371□4	1.2	0	47	47	0.0932	0
30□103	49013	48993.04545	0.04	0	43	37	0.0456	0.0306
508251	132875	131451.6889	1.1	0	202	115	0.3273	0.0466
508252	104173	103615.8862	0.53	0	112	109	0.2958	0.0616
508253	39028	38161.45919	2.2	0	117	87	0.3584	0.0776
510251	123694	123430.125	0.21	0	135	112	0.3898	0.0926
510252	104612	102067.9776	2.4	0	121	86	0.468	0.1086
510253	165860	165385.6358	0.3	0	133	97	0.4838	0.1246
516251	242194	240456.9255	0.72	0	188	139	0.702	0.1406
516252	139364	137077.3121	1.64	0	171	99	0.7016	0.1566
516253	82667	81586.36069	1.31	0	186	152	0.7952	0.2032

Average duality gap 1%

Table 3c: Results from solving the instances by Branch and Bound only for Z_{ff4}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	64116	59828.44423	6.7	0	42	38	0.0632	0
305102	83841	80727.63726	3.7	0	44	43	0.0□3	0
305103	49013	45364.33803	7.4	0	48	30	0.1252	0.0316
508251	132875	128204.6241	3.5	0	139	77	0.1562	0.0466
508252	104173	98830.3783	5.1	0	88	81	0.1872	0.0626
508253	39028	34269.78501	12.2	0	133	59	0.2192	0.047
510251	123694	120437.5971	2.6	0	136	90	0.3436	0.0786
510252	104612	94084.80585	10.1	0	108	65	0.3278	0.063
510253	165860	162150.7091	2.2	0	111	80	0.3432	0.063
516251	242194	238752.7811	1.4	0	161	107	0.437	0.078
516252	139364	129452.4162	7.1	0	155	88	0.484	0.1096
516253	82667	77272.31793	6.5	0	129	80	0.6044	0.1246

Average duality gap 5.7%

Table 3d: Results from solving the instances by Branch and Bound only for Z_{ff5}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	4452	1474.197807	66.9	0	18	22	0.0306	0.015
305102	5365	3146.201088	41.4	0	37	31	0.0782	0.0466
305103	6265	2365.17827	62.3	0	37	22	0.1092	0.0626
508251	9475	6763.782251	28.6	0	162	63	0.1568	0.0776
508252	8032	6382.212178	20.5	0	100	66	0.1722	0.0936
508253	6328	4040.014117	36.2	0	128	94	0.2032	0.1086
510251	10095	5602.913232	44.5	0	237	63	0.2804	0.093
510252	10839	4829.453995	55.4	0	182	78	0.4052	0.093
510253	9655	6734.395924	30.3	0	124	85	0.4054	0.093
516251	11654	8117.376639	30.4	0	264	84	0.7948	0.1236
516252	10054	5440.059974	45.9	0	280	89	0.6862	0.1396
516253	8249	5450.517218	33.9	0	272	101	0.718	0.1556

Average duality gap 41%

Table 3e: Results from solving the instances by Branch and Bound only for Z_{ff6}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	55152.8619	52855.61174	4.2	0	103	49	0.0938	0.0306
305102	67061.31787	66420.58656	1.	0	35	49	0.0942	0.0466
305103	36575.5455	36077.40661	1.4	0	46	37	0.0946	0.031
508251	63392.8448	62834.96705	1.0	0	136	120	0.1262	0.0616
508252	53793.94231	53396.96705	1.0	0	124	102	0.1266	0.0776
508253	22503.2	22380.84869	5.4	0	77	94	0.1892	0.0926
510251	62129.54196	61280.50053	1.4	0	162	126	0.4386	0.1086
510252	43234.99827	42595.93414	1.5	0	149	113	0.3754	0.1246
510253	109651.0481	109072.8718	1.0	0	133	131	0.3908	0.124
516251	135244.858	134626.9252	0.5	0	158	182	0.5006	0.1556
516252	77216.70671	76883.20928	0.43	0	163	130	0.4848	0.1716
516253	28860.38889	26794.82175	7.2	0	230	186	0.563	0.1876

Average duality gap 1.7%

Table 3f: Results from solving the instances by Branch and Bound only for Z_{ff7}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	9784	7670.846455	21.6	0	21	30	0.0938	0.016
305102	49719	47980.98624	3.5	0	25	27	0.1□92	0.031
305103	31205	27964.32□3	10.4	0	21	26	0.1086	0.0626
508251	47249.6667	44214.65511	6.4	0	88	70	0.1562	0.047
508252	41797.2222	38568.68722	7.7	0	101	81	0.1872	0.047
508253	17602	15514.14131	11.9	0	51	73	0.1866	0.047
510251	23952	21987.65612	8.2	0	51	75	0.203	0.063
510252	26752	23554.07376	12.0	0	99	75	0.4994	0.0936
510253	30641	29607.14141	3.4	0	64	96	0.3118	0.1096
516251	90652.9412	88832.24385	2.0	0	104	115	0.3584	0.11
516252	28433	26916.53454	5.3	0	98	107	0.3894	0.1406
516253	13958	12540.75158	10.2	0	94	128	0.3892	0.1566

Average duality gap 8.5%

Table 4a: Results from solving the instances by Branch and Bound and representation of fcc for Z_{ff1}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	7650.04761	5230.304569	32	0	135	51	0.156	0.0316
305102	8844.82529	6438.293597	27	0	126	55	0.1718	0.0466
305103	7543	4302.187724	43	0	106	60	0.2588	0.047
508251	14244.0083	11608.26248	19	0	818	97	0.608	0.0786
508252	11214.57500	9761.603636	13	0	518	123	0.7804	0.0946
508253	7931.66667	6050.060965	24	0	491	161	0.827	0.094
510251	14412.19805	10658.5035	26	0	733	134	1.202	0.1256
510252	14493.58876	10282.44702	29	0	845	168	1.4044	0.1566
510253	16972.3169	13050.25395	23	0	809	135	1.6078	0.156
516251	18353.06312	15412.14717	16	0	540	143	1.8878	0.1876
516252	15365.67724	11719.90232	24	207	5612	206	2.044	0.2036
516253	12174.5000	9047.522327	26	39	2812	261	3.7602	0.2196

Average duality gap 25%

Table 4b: Results from solving the instances by Branch and Bound and representation of fcc for Z_{ff3}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	4452	1728.853261	61	0	30	28	0.0622	0.0316
305102	5365	3369.668585	37	0	69	45	0.0932	0.0466
305103	6265	2577.293478	59	0	68	33	0.1242	0.031
508251	9475	6930.892982	27	0	152	100	1.091	0.0616
508252	8032	6395.639842	20	0	99	89	0.1866	0.0776
508253	6328	4311.268014	32	0	128	104	0.2026	0.078
510251	10095	5808.680597	42	0	293	69	0.3586	0.093
510252	10839	5092.290368	53	0	322	81	0.562	0.1236
510253	9655	6858.432486	29	0	142	85	0.578	0.28
516251	11654	8123.380582	30	0	263	112	0.546	0.1546
516252	10054	5953.360469	41	0	312	114	0.7024	0.155
516253	8249	5840.833543	29	0	237	171	1.56	0.1866

Average duality gap 38%

Table 4c: Results from solving the instances by Branch and Bound and representation of fcc for Z_{ff4}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	4452	1474.197807	67	0	2	22	0.0778	0
305102	5365	3146.201088	41	0	82	34	0.0932	0
305103	6265	2314.968781	63	0	37	25	0.0936	0.0306
508251	9475	6686.772753	29	0	173	62	0.1402	0.3256
508252	8032	6026.539993	25	0	135	67	0.1396	0.0616
508253	6328	3764.034875	41	0	111	62	0.1556	0.0776
510251	10095	5597.761304	45	0	564	60	0.3422	0.0936
510252	10839	4616.915084	57	0	362	63	0.3584	0.093
510253	9655	6616.727882	31	0	121	71	0.3276	0.093
516251	11654	8038.171642	31	0	434	94	0.4836	0.093
516252	10054	5350.673335	47	0	803	97	0.6554	0.093
516253	8249	5200.970236	37	0	2812	80	3.7602	0.093

Average duality gap 43%

Table 4d: Results from solving the instances by Branch and Bound and representation of fcc for Z_{ff5}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	8484	6094.9532	28	0	87	53	0.1098	0.016
305102	8844.8253	7633.3905	14	0	140	51	0.1558	0.0476
305103	7543	5006.1540	34	0	95	40	0.1712	0.032
508251	15228.71481	13637.5375	10	0	521	104	0.5148	0.048
508252	13442.31765	11187.13885	17	0	448	80	0.6236	0.0786
508253	7931.66667	6361.0554	20	0	295	121	0.5924	0.0946
510251	14412.19798	12491.4331	13	0	558	128	0.9198	0.1106
510252	14493.58876	12320.41284	15	0	390	143	1.0446	0.11
510253	17764.3354	15334.62087	14	0	596	112	1.2162	0.126
516251	20652.15877	18268.53508	12	0	334	111	1.4964	0.1576
516252	15888.78125	13860.49256	13	0	697	159	1.7774	0.1736
516253	12174.5000	10448.2797	14	0	770	153	1.9954	0.2046

Average duality gap 17%

Table 4e: Results from solving the instances by Branch and Bound and representation of fcc for Z_{ff7}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	4452	1474.1978	67	0	0	22	0.0782	0
305102	5365	3146.2011	41	0	38	34	0.0786	0.015
305103	6265	2314.9688	63	0	3	25	0.135	0.0466
508251	9475	6686.7728	29	0	130	62	0.1412	0.0616
508252	8032	6026.5399	25	0	106	67	0.235	0.218
508253	6328	3764.0349	41	0	116	62	0.2042	0.062
510251	10095	5597.7613	45	0	496	60	0.329	0.062
510252	10839	4616.9151	57	0	441	61	0.4226	0.062
510253	9655	6616.7279	31	0	131	71	0.3912	0.062
516251	11654	8038.1716	31	0	473	94	0.5632	0.077
516252	10054	5350.6733	47	0	764	96	0.9062	0.077
516253	8249	5200.9702	37	0	404	80	0.7348	0.077

Average duality gap 43%

Table 5: Results from solving the instances by Branch and Bound only for Z_{mcf}

problem	$z(.)$	$\bar{z}(.)$	ξ_{bound}	# B&B	# MIP Iterations	# dual simplex iterations	Time MIP	Time LP
305101	55152.8619	50989.75976	7.5	0	71	61	0.3116	0.015
305102	67061.31787	65703.9948	2.0	0	52	40	0.2178	0.046
305103	36575.5455	36077.40661	1.4	0	29	21	0.312	0.0625
508251	63392.8448	60436.17996	4.7	0	316	88	0.5622	0.0786
508252	53793.94231	52077.31623	3.2	0	124	96	0.515	0.0926
508253	22503.2	21547.18779	4.2	0	84	72	0.5614	0.1086
510251	62129.54196	59718.91831	3.9	0	318	135	0.8574	0.1552
510252	43234.99827	41043.9269	5.1	0	208	105	0.9664	0.1706
510253	109651.0481	107150.7403	2.3	0	300	192	0.9974	0.2016
516251	135244.858	134032	0.	0	186	151	1.544	0.2482
516252	77216.70671	72350.31579	6.3	0	354	173	1.6226	0.263
516253	28860.38889	24287.14113	1.6	0	210	147	1.7948	0.3112

Average duality gap 4.78%

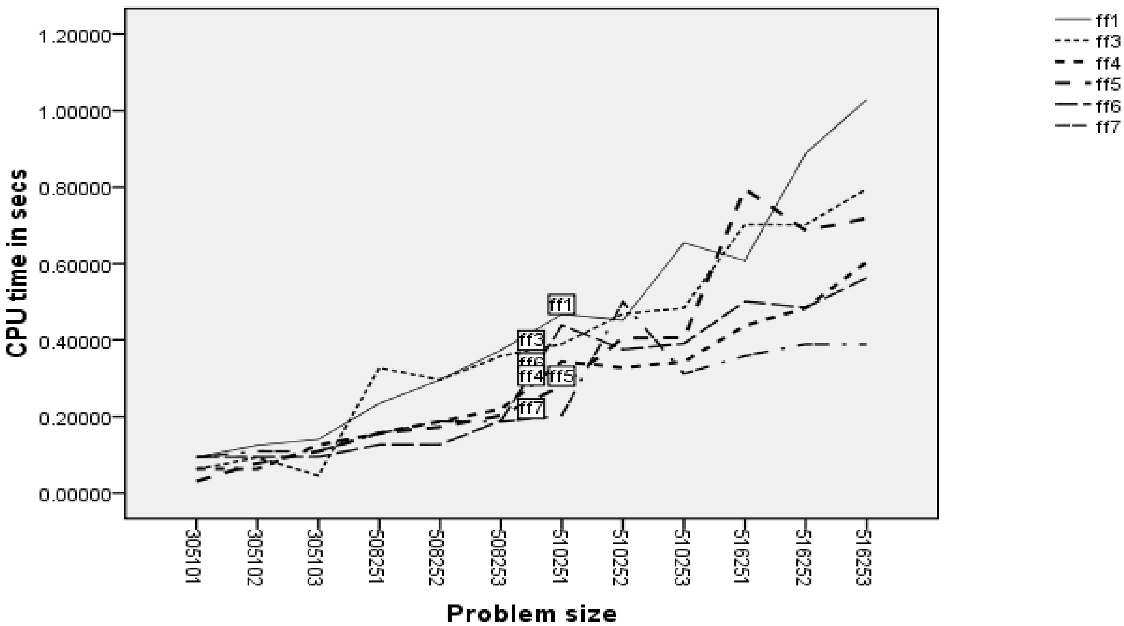

Figure 1. Graph of MIP CPU time for all flow formulations

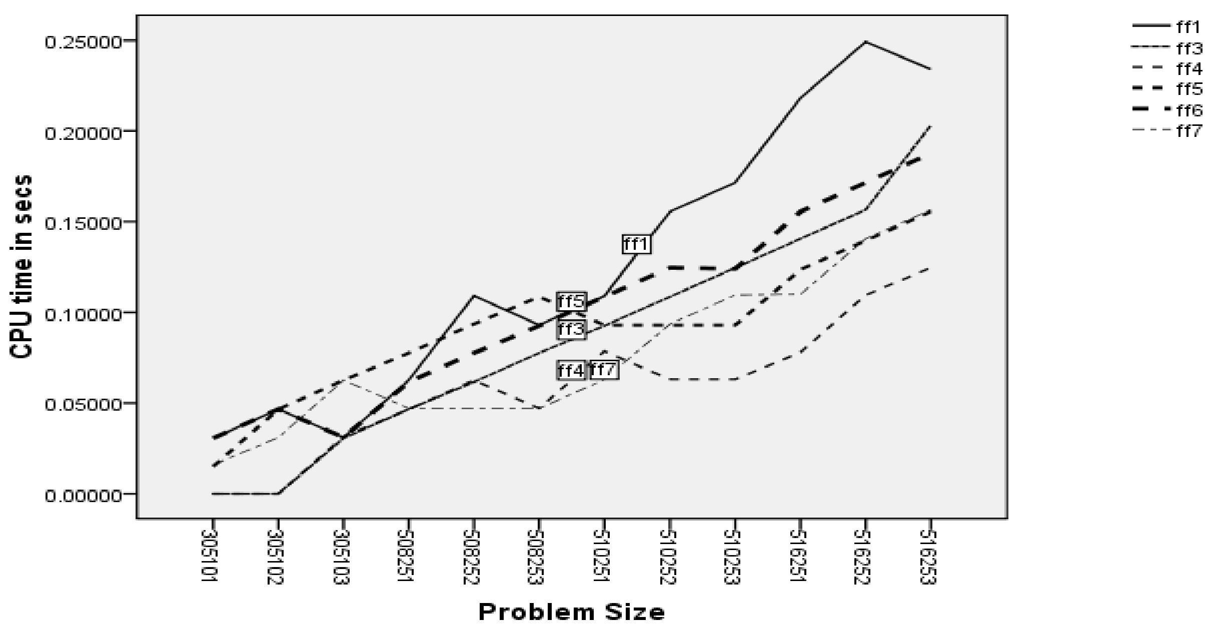

Figure 2. Graph of LP CPU time for all flow formulations

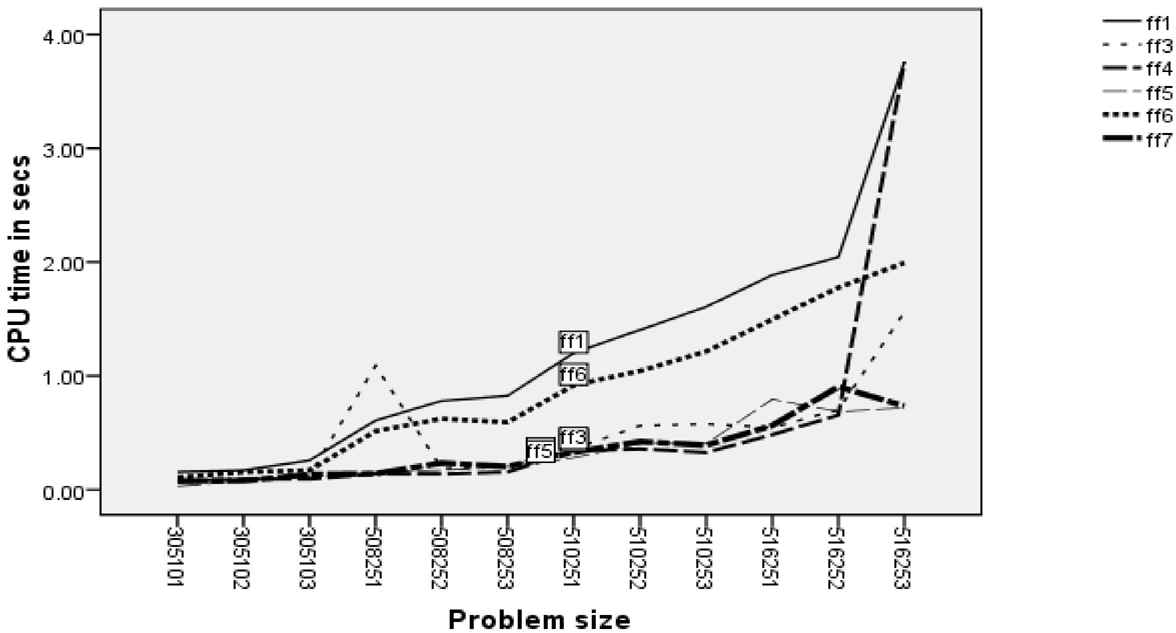

Figure 3.Graph of MIP CPU time, fcc for all flow formulations

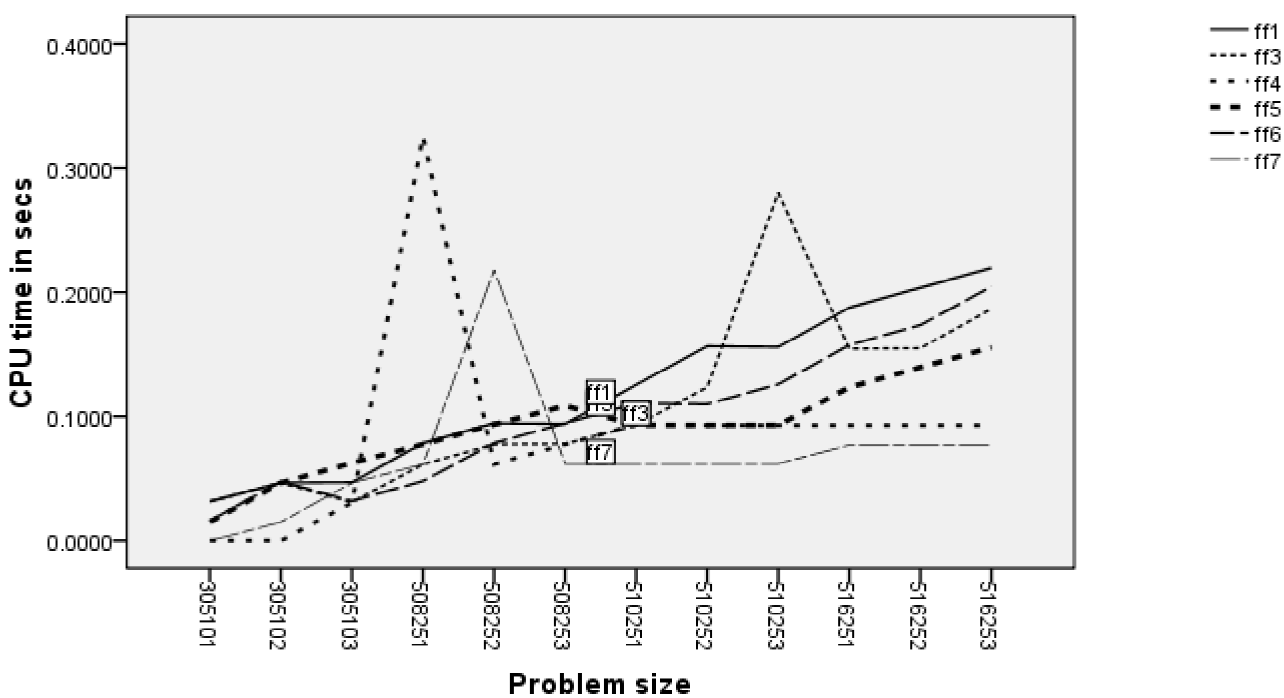

Figure 4. graph of LP CPU time, fcc for all flow formulations

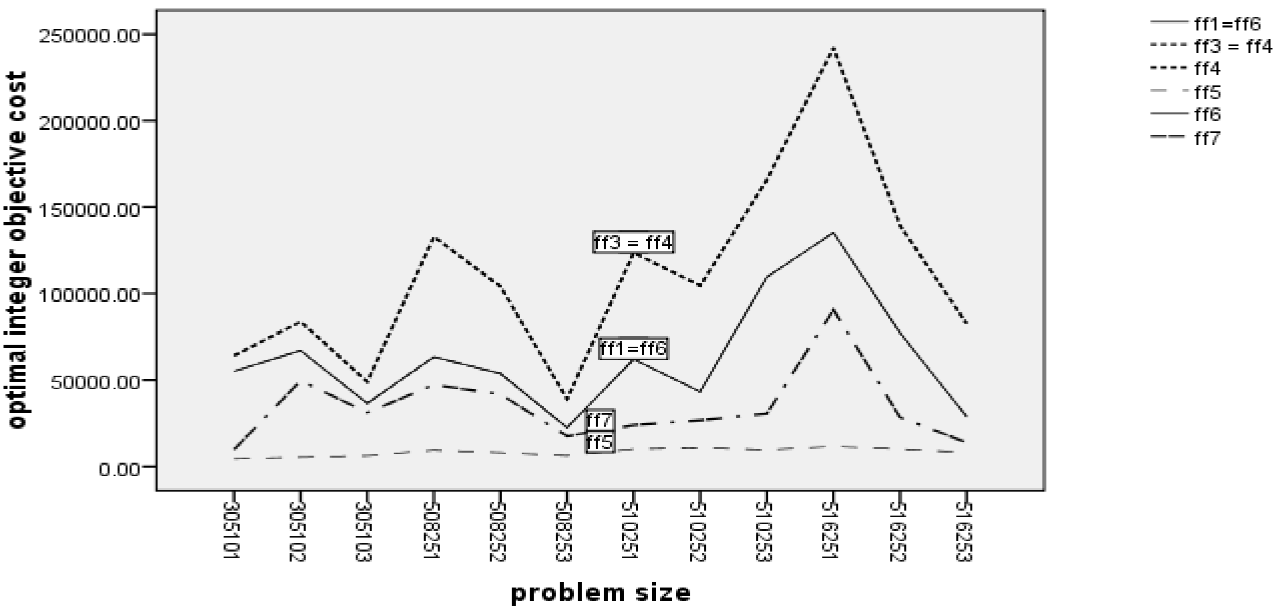

Figure 5. Graph of optimal integer objective costs for all flow formulations

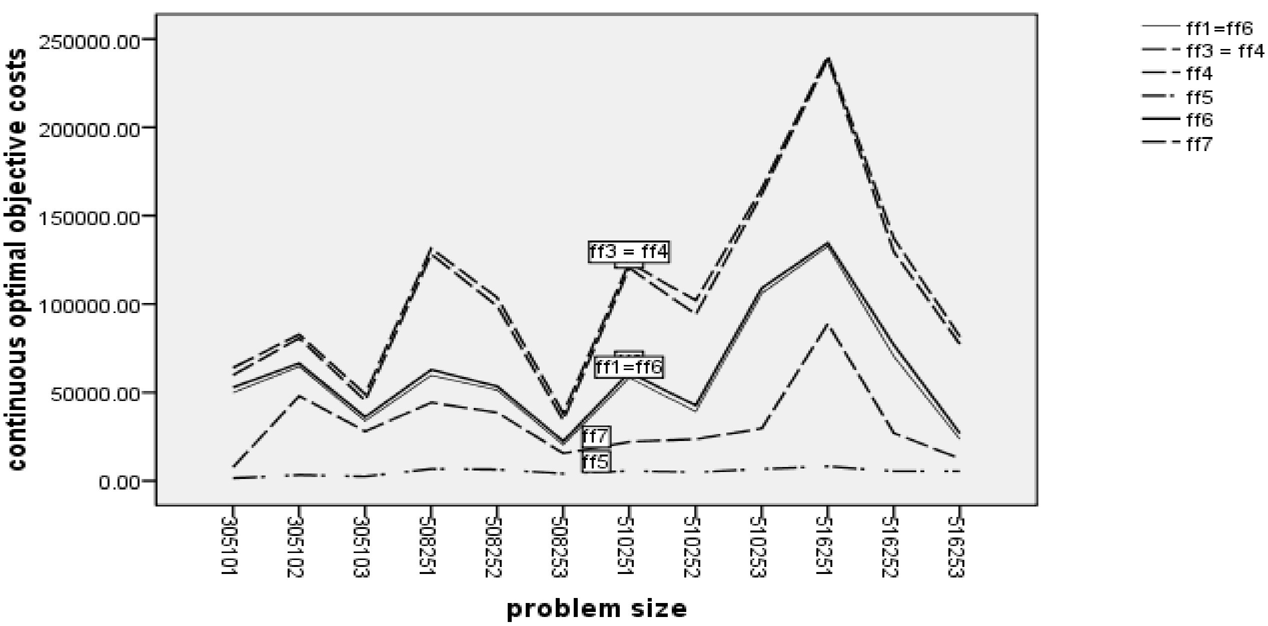

Figure 6. Graph of optimal LP objective costs for all flow formulations

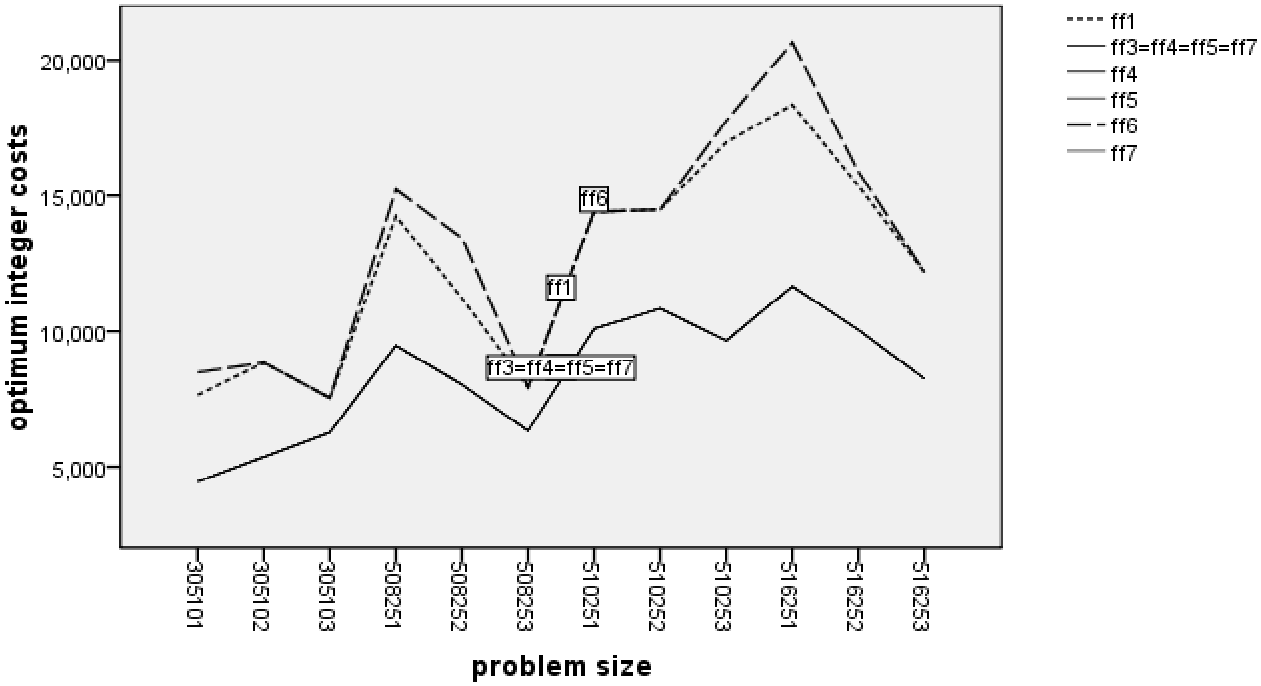

Figure 7. Graph of optimal integer objective costs for B&B and fcc

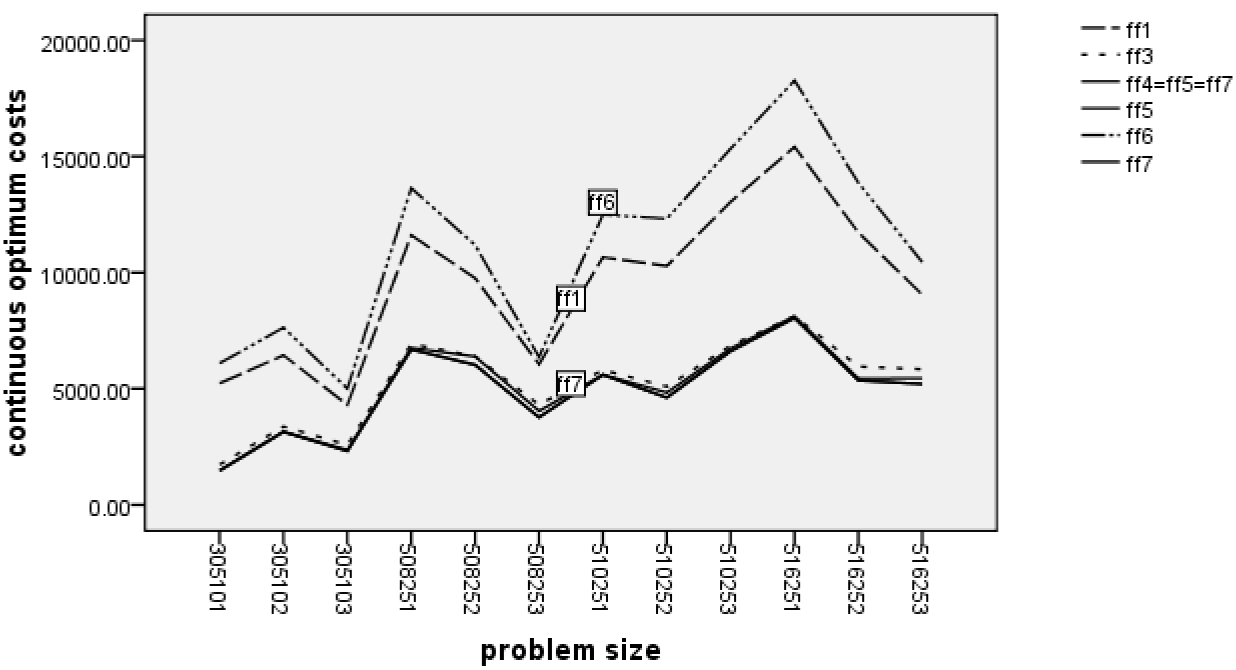

Figure 8. Graph of LP optimal objective costs for B&B and fcc

5. Conclusion

In this paper we study alternative formulations of two stage capacitated facility location (TSCFL) problem. We have shown that the two obvious ways of formulating this problem, i.e. the flow and multi commodity formulations are equivalent. From computational point of view we conclude among others that:

- The flow conservation constraints when expressed correctly tighten the formulation and yield a better lower bound as well as the optimal integer objective value than when left otherwise (see Figures 5, 6, 7 and 8).
- Adding the valid inequalities as well as expressing the flow conservation constraint correctly tightens the formulation, but unfortunately this introduces higher duality gap in the solutions.
- The LP-relaxation of the multi commodity formulation appeared to be better than that of the flow formulation of TSCFL problem, when the instances were solved by branch-and-bound only.
- Among the equivalent formulations, the graphs of various parameters, such as CPU time for MIP and LP-relaxation, optimal integer as well as continuous costs of the objective function further highlight the dominant relationship that exists among the formulations in terms of computing resources.
- From computational point of view, from the tables and graphs we can conclude further that Z_{ff7} perform better than the rest, when the instances were solved by branch and bound only. While Z_{ff5}, Z_{ff7} and Z_{ff4} performs better when the instances were solved by branch and bound and expressing correctly the flow conservation constraints.
- The main objective of any mathematical programming problem is to optimize (i.e. maximize or minimize) an objective function subject to certain constraints. Computationally Figures 5, 6, 7, and 8 suggest that we recommend models formulation Z_{ff5} and Z_{ff7}, for TSCFLP due to their respective minimum objective costs.

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