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Gravitational Field Equations Exterior to Rotating Homogeneous Spherical Mass Distributions

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Abstract

Gravitational field equations exterior to rotating homogeneous spherical masses are derived. The field equations have only one unknown function dependent on the mass or pressure distribution. In approximate gravitational fields, the unknown function reduces to the gravitational scalar potential exterior to the mass. Thus, this approach introduces a hither to unknown way of generalizing the gravitational scalar potential and introduces correction terms to it. In this gravitational field, the sixteen field equations reduce to four nonlinear partial differential equations.

1.0 Introduction

In recent articles [1-4], we showed how gravitational fields of various mass distributions can be studied using metric tensors that have only one arbitrary function determined by the mass or pressure distribution. This arbitrary function possesses all the symmetries of the mass or pressure distribution. In [1], we obtained the metric tensor for a rotating homogeneous spherical mass and used it to study the motion of test particles and photons in this gravitational field. Orbits in the vicinity of a rotating homogeneous mass were also studied. In [2], we constructed gravitational field equations for a time varying spherical mass distribution and obtained gravitational radiation equations in this gravitational field. In [3], the metric tensor in the field of an oblate spheroidal mass was constructed and used to study orbits in the vicinity of an oblate spheroidal mass. In [4], gravitational spectral shift exterior to oblate spheroidal masses in the Solar System was studied using this new approach. In this article, we extend our work in [1], to derive Einstein's gravitational field equations exterior to rotating homogeneous spherical masses.

In [1], a static sphere of total mass M and density ρ placed in empty space is considered. In such a sphere, the mass distribution within the sphere is homogeneous and made to rotate with uniform angular velocity about a fixed diameter. It was shown that the covariant metric tensor for this gravitational field is given in spherical polar coordinates (r,θ,ϕ) as

$$g_{00} = 1 + \frac{2}{c^2} f(r,\theta) \tag{1.1}$$

$$g_{11} = -\left[1 + \frac{2}{c^2} f(r,\theta)\right]^{-1} \tag{1.2}$$

$$g_{22} = -r^2 \tag{1.3}$$

$$g_{33} = -r^2 \sin^2\theta \tag{1.4}$$

$$g_{\mu\nu} = 0; \quad otherwise \tag{1.5}$$

where $f(r,\theta)$ is an arbitrary function determined by the mass distribution within the sphere.

In this article, the metric tensor, equations (1.1) to (1.5) is used to construct gravitational field equations for this gravitational field.

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Using tensor analysis [5], the contravariant metric tensor and coefficients of affine connection can be constructed [1]. The Riemann Christoffel tensor (in terms of the affine connection coefficients) for this gravitational field is found to have the following components

$$R^0_{012} = -R^0_{021} = \Gamma^0_{02,1} - \Gamma^0_{01,2} \tag{2.1}$$

$$R^0_{101} = -R^0_{110} = -\Gamma^0_{10,1} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^2_{11}\Gamma^0_{20} - \Gamma^0_{10}\Gamma^0_{01} \tag{2.2}$$

$$R^0_{102} = -R^0_{120} = -\Gamma^0_{10,2} + \Gamma^1_{12}\Gamma^0_{10} + \Gamma^2_{12}\Gamma^0_{20} - \Gamma^0_{10}\Gamma^0_{02} \tag{2.3}$$

$$R^0_{201} = -R^0_{210} = -\Gamma^0_{20,1} + \Gamma^1_{21}\Gamma^0_{10} + \Gamma^2_{21}\Gamma^0_{20} - \Gamma^0_{20}\Gamma^0_{01} \tag{2.4}$$

$$R^0_{202} = -R^0_{220} = -\Gamma^0_{20,2} + \Gamma^1_{22}\Gamma^0_{10} - \Gamma^0_{20}\Gamma^0_{02} \tag{2.5}$$

$$R^0_{303} = -R^0_{330} = \Gamma^1_{33}\Gamma^0_{10} - \Gamma^2_{33}\Gamma^0_{02} \tag{2.6}$$

$$R^1_{001} = -R^1_{010} = -\Gamma^1_{00,1} + \Gamma^0_{01}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^1_{11} - \Gamma^2_{00}\Gamma^1_{21} \tag{2.7}$$

$$R^1_{002} = -R^1_{020} = -\Gamma^1_{00,2} + \Gamma^0_{02}\Gamma^1_{00} - \Gamma^1_{00}\Gamma^1_{12} - \Gamma^2_{00}\Gamma^1_{22} \tag{2.8}$$

$$R^1_{112} = -R^1_{121} = \Gamma^1_{12,1} - \Gamma^1_{11,2} + \Gamma^2_{12}\Gamma^1_{21} - \Gamma^2_{11}\Gamma^1_{22} \tag{2.9}$$

$$R^1_{212} = -R^1_{221} = \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^1_{21} - \Gamma^1_{21}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^1_{22} \tag{2.10}$$

$$R^1_{313} = -R^1_{331} = \Gamma^1_{33,1} + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^2_{33}\Gamma^1_{21} - \Gamma^3_{31}\Gamma^1_{33} \tag{2.11}$$

$$R^1_{323} = -R^1_{332} = \Gamma^1_{33,2} + \Gamma^1_{33}\Gamma^1_{12} + \Gamma^2_{33}\Gamma^1_{22} - \Gamma^3_{32}\Gamma^1_{33} \tag{2.12}$$

$$R^2_{001} = -R^2_{010} = -\Gamma^2_{00,1} + \Gamma^0_{01}\Gamma^2_{00} - \Gamma^1_{00}\Gamma^2_{11} - \Gamma^2_{00}\Gamma^2_{21} \tag{2.13}$$

$$R^2_{002} = -R^2_{020} = -\Gamma^2_{00,2} + \Gamma^0_{02}\Gamma^2_{00} - \Gamma^1_{00}\Gamma^2_{12} \tag{2.14}$$

$$R^2_{112} = -R^2_{121} = -\Gamma^2_{12,1} - \Gamma^2_{11,2} + \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{21} - \Gamma^1_{11}\Gamma^2_{12} \tag{2.15}$$

$$R^2_{212} = -R^2_{221} = -\Gamma^2_{12,2} + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{21}\Gamma^2_{12} \tag{2.16}$$

$$R^2_{313} = -R^2_{331} = \Gamma^2_{33,1} + \Gamma^1_{33}\Gamma^2_{11} + \Gamma^2_{33}\Gamma^2_{21} - \Gamma^3_{31}\Gamma^2_{33} \tag{2.17}$$

$$R^2_{323} = -R^2_{332} = \Gamma^2_{33,2} + \Gamma^1_{33}\Gamma^2_{12} - \Gamma^3_{32}\Gamma^2_{33} \tag{2.18}$$

$$R^3_{003} = -R^3_{030} = -\Gamma^1_{00}\Gamma^3_{13} - \Gamma^2_{00}\Gamma^3_{23} \tag{2.19}$$

$$R^3_{113} = -R^3_{131} = \Gamma^3_{13,1} + \Gamma^3_{13}\Gamma^3_{31} - \Gamma^1_{11}\Gamma^3_{13} - \Gamma^2_{11}\Gamma^3_{23} \tag{2.20}$$

$$R^3_{123} = -R^3_{132} = \Gamma^3_{13}\Gamma^3_{32} - \Gamma^1_{12}\Gamma^3_{13} - \Gamma^2_{12}\Gamma^3_{23} \tag{2.21}$$

$$R^3_{213} = -R^3_{231} = \Gamma^3_{23}\Gamma^3_{31} - \Gamma^1_{12}\Gamma^3_{13} - \Gamma^2_{12}\Gamma^3_{23} \tag{2.22}$$

$$R^3_{223} = -R^3_{232} = \Gamma^3_{23,2} + \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13} \tag{2.23}$$

$$R^\alpha_{\beta\delta\phi} = 0; \quad otherwise \tag{2.24}$$

where $(0,1,2,3)$ represent the (ct, r, θ, ϕ) four space-time coordinates.

From the Riemann Christoffel tensorequations (2.1) to (2.24) and tensor analysis, the Ricci tensor for this gravitational field is obtained as

$$R_{00} = R^0_{000} + R^1_{001} + R^2_{002} + R^3_{003} \tag{2.25}$$

$$R_{11} = R^0_{110} + R^1_{111} + R^2_{112} + R^3_{113} \tag{2.26}$$

$$R_{22} = R^0_{220} + R^1_{221} + R^2_{222} + R^3_{223} \tag{2.27}$$

$$R_{33} = R^0_{330} + R^1_{331} + R^2_{332} + R^3_{333} \tag{2.28}$$

$$R_{\mu\nu} = 0; \quad otherwise \tag{2.29}$$

Substituting explicit expressions for the components of the curvature tensors and simplifying yields the following explicit expressions for the Ricci tensor;

$$R_{00} = -\frac{1}{c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]\frac{\partial^2 f}{\partial r^2} - \frac{2}{rc^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]\frac{\partial f}{\partial r}$$
$$-\frac{\cot\theta}{r^2c^2}\frac{\partial f}{\partial r} - \frac{1}{r^2c^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{2}{r^2c^4}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\left(\frac{\partial f}{\partial\theta}\right)^2 \tag{2.30}$$

$$R_{11} = \frac{1}{c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\frac{\partial^2 f}{\partial r^2} + \frac{2}{rc^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\frac{\partial f}{\partial r} - \frac{\cot\theta}{r^2c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-2}\frac{\partial f}{\partial\theta}$$
$$-\frac{1}{r^2c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-2}\frac{\partial^2 f}{\partial\theta^2} + \frac{2}{r^2c^4}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-3}\left(\frac{\partial f}{\partial\theta}\right)^2 \tag{2.31}$$

$$R_{22} = \frac{2}{c^4}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-2}\left(\frac{\partial f}{\partial\theta}\right)^2 + \frac{2r}{c^2}\frac{\partial f}{\partial r} + \frac{2}{c^2}f(r,\theta) \tag{2.32}$$

$$R_{33} = \frac{3r\sin^2\theta}{c^2}\frac{\partial f}{\partial r} + \frac{2\sin^2\theta}{c^2}f(r,\theta) + \cos 2\theta \tag{2.33}$$

From, the Ricci tensor, the curvature scalar, R, for this gravitational field is obtained using

$$R = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \tag{2.34}$$

as

$$R = -\frac{2}{c^2}\frac{\partial^2 f}{\partial r^2} - \frac{8}{rc^2}\frac{\partial f}{\partial r} - \frac{\cot\theta}{r^2c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\frac{\partial f}{\partial r} - \frac{2}{r^2c^4}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-2}\left(\frac{\partial f}{\partial\theta}\right)^2$$
$$+\frac{\cot\theta}{r^2c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\frac{\partial f}{\partial\theta} - \frac{4}{r^2c^2}f(r,\theta) - \frac{1}{r^2}\frac{\cos 2\theta}{\sin^2\theta} \tag{2.35}$$

It can be easily deduced from the Ricci tensor that the sixteen general relativistic field equations exterior to the mass distribution reduce to four equations in this gravitational field. These are:

$$R_{00} - \frac{1}{2}Rg_{00} = 0 \tag{2.36}$$

$$R_{11} - \frac{1}{2}Rg_{11} = 0 \tag{2.37}$$

$$R_{22} - \frac{1}{2}Rg_{22} = 0 \tag{2.38}$$

$$R_{33} - \frac{1}{2}Rg_{33} = 0 \tag{2.39}$$

Substituting the Ricci tensor, curvature scalar and the covariant metric tensor into the field equations (2.36 - 2.39) and simplifying yields the respective explicit field equations as:

$$0 = \frac{\partial^2 f}{\partial\theta^2} - \frac{3}{c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\left(\frac{\partial f}{\partial\theta}\right)^2 + \frac{\cot\theta}{2}\frac{\partial f}{\partial\theta} - 2r\left[1+\frac{2}{c^2}f(r,\theta)\right]\frac{\partial f}{\partial r}$$
$$+\frac{\cot\theta}{2}\frac{\partial f}{\partial r} - 2\left[1+\frac{2}{c^2}f(r,\theta)\right]f(r,\theta) - \frac{c^2}{2}\frac{\cos 2\theta}{\sin^2\theta}\left[1+\frac{2}{c^2}f(r,\theta)\right] \tag{2.40}$$

$$0 = \frac{\partial^2 f}{\partial\theta^2} - \frac{1}{c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\frac{\partial f}{\partial\theta} + \frac{\cot\theta}{2}\frac{\partial f}{\partial\theta} + 2r\left[1+\frac{2}{c^2}f(r,\theta)\right]\frac{\partial f}{\partial r}$$
$$+2\left[1+\frac{2}{c^2}f(r,\theta)\right]f(r,\theta) + 2c^2\frac{\cos 2\theta}{\sin^2\theta}\left[1+\frac{2}{c^2}f(r,\theta)\right] \tag{2.41}$$

$$0 = \frac{\partial^2 f}{\partial r^2} + \frac{3}{r^2c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-2}\left(\frac{\partial f}{\partial\theta}\right)^2 + \frac{\cot\theta}{2r^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial\theta}\right)$$
$$+\frac{c^2}{2r^2}\frac{\cos 2\theta}{\sin^2\theta} + \frac{6}{r}\frac{\partial f}{\partial r} + \frac{4}{c^2}f(r,\theta) \tag{2.42}$$

$$0=\frac{\partial^2 f}{\partial r^2}-\frac{1}{r}\frac{\partial f}{\partial r}+\frac{\cot\theta}{r^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-1}\left(\frac{\partial f}{\partial r}-\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2c^2}\left[1+\frac{2}{c^2}f(r,\theta)\right]^{-2}\left(\frac{\partial f}{\partial\theta}\right)^2 \tag{2.43}$$

In weak gravitational fields (to the order of c^{-2}), these field equations reduce respectively to

$$0=\frac{\partial^2 f}{\partial\theta^2}-\frac{3}{c^2}\left(\frac{\partial f}{\partial\theta}\right)^2+\frac{\cot\theta}{2}\left(\frac{\partial f}{\partial\theta}+\frac{\partial f}{\partial r}\right)-2r\left[1+\frac{2}{c^2}f(r,\theta)\right]\frac{\partial f}{\partial r}$$
$$-2\left[1+\frac{2}{c^2}f(r,\theta)\right]f(r,\theta)-\frac{c^2}{2}\frac{\cos 2\theta}{\sin^2\theta}\left[1+\frac{2}{c^2}f(r,\theta)\right] \tag{2.44}$$

$$0=\frac{\partial^2 f}{\partial\theta^2}-\frac{1}{c^2}\frac{\partial f}{\partial\theta}+\frac{\cot\theta}{2}\frac{\partial f}{\partial\theta}+2r\left[1+\frac{2}{c^2}f(r,\theta)\right]\frac{\partial f}{\partial r}$$
$$+2\left[1+\frac{2}{c^2}f(r,\theta)\right]f(r,\theta)+2c^2\frac{\cos 2\theta}{\sin^2\theta}\left[1+\frac{2}{c^2}f(r,\theta)\right] \tag{2.45}$$

$$0=\frac{\partial^2 f}{\partial r^2}+\frac{3}{r^2c^2}\left(\frac{\partial f}{\partial\theta}\right)^2+\frac{\cot\theta}{2r^2}\left[1-\frac{2}{c^2}f(r,\theta)\right]\left(\frac{\partial f}{\partial r}-\frac{\partial f}{\partial\theta}\right)$$
$$+\frac{c^2}{2r^2}\frac{\cos 2\theta}{\sin^2\theta}+\frac{6}{r}\frac{\partial f}{\partial r}+\frac{4}{c^2}f(r,\theta) \tag{2.46}$$

$$0=\frac{\partial^2 f}{\partial r^2}-\frac{1}{r}\frac{\partial f}{\partial r}+\frac{\cot\theta}{r^2}\left[1-\frac{2}{c^2}f(r,\theta)\right]\left(\frac{\partial f}{\partial r}-\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2c^2}\left(\frac{\partial f}{\partial\theta}\right)^2 \tag{2.47}$$

The gravitational field equations reduce to four nonlinear partial differential equations that can be solved to obtain explicit expressions for the unknown function $f(r,\theta)$. It is well known [6], that for this mass distribution, the gravitational scalar potential $\Phi(r,\theta)$ is given as

$$g_{00}\approx 1+\frac{2}{c^2}\Phi(r,\theta) \tag{2.48}$$

Thus, our unknown function can be conveniently equated to the gravitational scalar potential exterior to the mass distribution. It is hoped that when the field equations are solved, hither to unknown generalization of the gravitational potential for this mass distribution will be obtained.

Conclusion

It is worth noting that the field equations constructed in this article satisfy the requirement of invariance of line element a priori and this is satisfactory for an astrophysical solution of Einstein's gravitational field. Also, the field equations are mathematically simple as they contain only one unknown. With the construction of gravitational field equations in this gravitational field, the door is thus open for the study of other gravitational phenomena in this field.

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