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Computation of Electric Fields by the Method of Successive Over-Relaxation

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Abstract

Using the method of successive over-relaxation the electric field computation is presented. We have restricted our result to the electric field in the vacuum where the solution of Laplace's equation is obtained. The numerical solution of Laplace's equation with a boundary values, specified, were used on a finite element with successive over-relaxation. The main program used was one in which the semi-separation distances of flanges d and the tube radius r were replaced by the dimensionless interger m and n in KAP9. In the program m_{max} and n_{max} were chosen to be sufficiently large enough from the start. The w is specified in such away that one obtains the fastest possible convergence. The short computation time was achieved by setting m = n = 5, ϕ_0 = 1000V and w =1.0, 1.2, 1.4, 1.6, 1.8, 1.9, and 1.99.

1.0 Introduction

The equations of motion for charged particles in electromagnetic fields can be solved by method of successive over-relaxation[1]. In this way one can, for example, study the focusing properties of guide fields in particle accelerators or the imaging properties of lenses in electron optics. The assumption here is that the electromagnetic fields are known, and this brings us to the next problem, namely, the boundary problem in partial differential equations. In what follows it will now be shown how one can, by means of the computer, enlarge the repertoire of examples with well known field patterns.

1.1 Theory

Using *electric* fields in the vacuum[2-7], i.e. we study solutions of Laplace's equations,

$$\Delta\Phi = 0 \tag{1}$$

As boundary condition we specify the potential on the boundary of the region in which the equation is to be solved is specified.As an example a simple electrostatic lens which, with slight modifications, is still used in electron optics. A cylindrical metal tube with radius r_0 is interrupted by flanges F_1, F_2, as shown in Fig. 1.

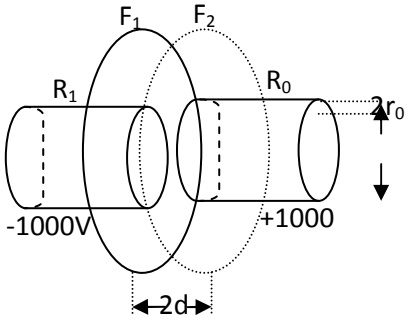

Fig. 1. Simple electrostatic lens containing a two pieces of tube R_1, R_2 and flanges F_1, F_2.

The flanges form a plate condenser with the two annular plates F_1, f_2,at a separation $2d$, to which the two pieces of tube R_1, R_2,are attached. Let the right-hand flange be charged to a potential of 1000 V, and the left to a potential of -1000V(In electron optics higher voltages are usually employed). To avoid field emission the corners of the tube/flange transitions have to be rounded off'; this is true even at 1000 V for very small values of d. In our calculation we shall not take account of this

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founding off of the corners. One seeks the form of the potential Φ (*x,y,z*)in the region between theplated and in the tubes. If the potential is known as a function of the position coordinates, then one obtains the electric field from

$$E = -\text{grad}\,\Phi \quad (2)$$

The solution of Lapace's equation (1) is uniquely determined in a closed volume, when the potential is specified over the surface of the volume. Up to now the volume in which we wish to compute the potential function is still open. Using a little physical intuition, however, we can close it. If we go far enough along the tube, then the potential in each tube is scarcely going to change any more. This means that, to the far right and the far left of the flanges we shall find $\Phi_1 = +1000\,V$ and $\Phi_2 = -\Phi_1 = -1000\,V$, respectively, in the tubes. In other words, the potential field will not be changed if we close the tubes far to the right and far to the left of the flanges with metal lids. Between the plates the potential field will be more and more similar to that of an ordinary plate condenser, the further we are from the axis of the tubes. This means that here also we can close the volume, only this time not with a metal plate but with an insulator or simply with an imaginary surface. On this surface the potential varies linearly with distance from the plates, from +1000V on the right-hand plate to -1000V on the left-hand plate. The potential is thereby specified on the surface of closed volume, and we can now consider how to solve Laplace's equation.

First we make use of the axial symmetry of the configuration[8-11]in order to reduce the number of coordinates from 3 to 2. We introduce cylindrical coordinate[12-14]. The axis of the tubes is the z-axis. The coordinates *x,* y are replaced by plane polar coordinates r, φ, where r denotes the distance from the z-axis and φ the azimuthal angle around the z-axis. In cylindrical coordinates Laplace's equation can be written as:

$$\Delta\Phi = \frac{\partial^2\Phi}{\partial z^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial r^2} + + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0, \quad (r > 0) \quad (3)$$

Because of the axial symmetry the potential Φ cannot depend on the azimuthal angle ϕ,and the equation simplifies to.

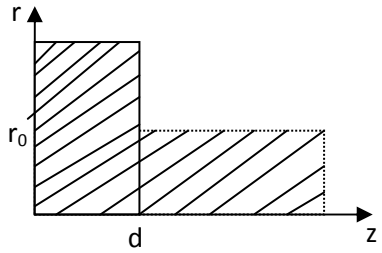

Fig. 2 Shows the region in which Laplace's equation is to be solve

$$\Delta\Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \right)\Phi(\mathrm{z},\mathrm{r}) = 0, \quad (r > 0) \quad (4)$$

Moreover, we see that the configuration is symmetric about the plane z = 0. It is therefore sufficient to solve (4) in the shaded region of Fig. 2. This region is still open below. However, since the lower edge is identical with the axis of symmetry, no boundary condition is needed here.

For the numerical solution of (4) with the boundary values specified, we usea finite element method with successive over-relaxation.

1.2 Numerical Method

For the numerical solution of Laplace's equation we shall approximate Φ *(*z,r) on a two-dimensional grid of points with constant mesh width *h.* Instead of the function Φ *(z,r)* we shall therefore consider only a matrix ($\Phi_{1,k}$) of function values [15]. Let

$$z_i = ih, \quad r_k = kh, \quad \Phi_{i,k} = \Phi(z_i r_k) \quad (5)$$

Since a very simple shape for the electrode boundariesischosen, it becomeseasy to fit these into the grid. We only have to choose *h* so that the tube radius $r_0$and the semi-distance separating the flanges *d* are integral multiples of h. More complicated shapes of electrode have to be represented by a polygonal approximation.

In order to solve Laplace's equation (4) on this grid, we have first to discretisethe differential operators $\partial/\partial_r\ \partial^2/\partial r^2$.The formulae $f'(z) = \dfrac{f(z+h) - f(z-h)}{2h} + 0(h^2)$ and $f'(z) = \dfrac{f(z+h) - f(z-h) - 2f(z)}{h^2} + 0(h^2)$ is used and obtained for r > 0:

$$\Delta\Phi = \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\Phi(\mathrm{z,r})$$

$$= \frac{1}{h^2}\left(\Phi(z, r+h) + \Phi(z, r-h) + \Phi(z+h, r) + \Phi(z-h, r) - 4\Phi(\mathrm{z,r})\right)$$

$$+ \frac{1}{r}\frac{1}{2h}(\Phi(z, r+h) - \Phi(z, r-h) + O(h^2) = 0 \qquad (6)$$

We now apply the relations (6), neglect the error term $O(\mathrm{h}^2)$, multiply the equation by h^2 and so obtain an approximation of Laplace's equation on the grid, for $k > 0$:

$$(\Phi_{\mathrm{i,k+1}} + \Phi_{\mathrm{i,k-1}} + \Phi_{\mathrm{i+1,k}} + \Phi_{\mathrm{i-1,k}} - 4\Phi_{\mathrm{i,k}}) + \frac{1}{2\mathrm{k}}\left(\Phi_{\mathrm{i,k+1}} - \Phi_{\mathrm{i,k-1}}\right) = 0 \qquad (7)$$

For $k = 0$, i.e. for the grid points on the axis of symmetry, (7) is not valid. For these points we have to formulate a special equation. The equation is most simply obtained by first converting back into Cartesian coordinates x,y,zand using($f'(z) = \frac{f(z+h) + f(z-h) - 2f(z)}{h^2} + 0(h^2)$):

$$\Delta\Phi(\mathrm{x,y,z}) = \frac{1}{h^2}(\Phi(x+h, y, z) + \Phi(x-h, y, z) + \Phi(x, y+h, z) + \Phi(x, y-h, z) + \Phi(x, y, z+h) + \Phi(x, y, z-h) - 6\Phi(\mathrm{x,y,z}) + 0(\mathrm{h}^2) \qquad (8)$$

We wish to apply this equation at the axis of symmetry and accordingly set $x = y = 0$. Now we return from the potential function Φ *(x,y,z)* in Cartesian coordinates to the potential function Φ *(z,r)* in cylindrical coordinates. The first four function values on the right-hand side are all equal, namely equal to Φ *(z,h).* If we again neglect $O(h^2)$, multiply throughout by h^2 and use the relations (5), we obtain the so-called axis formula:

$$4\Phi_{\mathrm{i,1}} + \Phi_{\mathrm{i+1,0}} - 6\Phi_{\mathrm{i,0}} = 0 \qquad (9)$$

1.3 The Method of Successive Over-relaxation

If one writes down (7) for all interior points of the grid and (9) for all points on the axis of symmetry (with the exception of the upper and lower boundary points), then one obtains a system of linear equations, from which in principle the matrix ($\Phi_{i,k}$) can be calculated. Since we are to calculate many function values $\Phi_{i,k}$, however, this system of equations has a high dimension, The usual methods for solving systems of linear equations, such as the Gaussian elimination method, are then no longer suitable for the solution of the problem.

The system of equations (7), (9), however, has one peculiarity: each point is linked only with the directly neighboring points. If we represent the Laplace operator as an enormous matrix, this matrix is almost empty: most of the elements are zero. For such problems the method of successive over-relaxation [16] has been developed.

The improved approximate solution at the point (i,k) is calculated by means of the formula

$$\Phi_{i,k}^{new} = \Phi_{i,k}^{old} + w(U - \Phi_{i,k}^{old}), \quad 1 \le \mathrm{w} < 2$$

The constant w is chosen so that one obtains the fates possible convergence.

2.0 programming

The method of successive over-relaxation can be programmed with little effort. The main program KAP9 [17] was used as the programmed.First of all we replace the semi-separation distance of the flanges d and the tube radius r^0 by the dimensionless integer *m* and *n*,

$$m = \frac{d}{h}, \quad n = \frac{r_0}{h}$$

The assumed wall which closes the end of the tube is at a distance $m_{max}$from the plane z=0, and the wall which encloses the flanges is at a distance$n_{max}$from the axis of symmetry. Thus,*h*is chosen as the unit of length and eliminated it from the calculation. m_{max}andn_{max}were specified. The solution will then show whether these values are too small and must be increased. In this program m_{max} *and*n_{max}were specifiedsufficiently large from the start.

In order to achieve a rather short computation time we set, m = n = 5, $\Phi_0 = 1000\mathrm{V}$ and *w* = 1.0, 1.2, 1.4, 1.6, 1.8, 1,9and 1.99,

3.0 Result

With *w*=1.0 (Fig 3) there is slow convergence. The largest change of thepotential value griddecreases monotonically from iteration to iteration. The series of largestchanges is similar to a geometrical series, i.e. with each iteration the largest change decreases by several percent. Qualitatively the same behaviour is found with *w* =1.2 (Fig 4) and w =1.4 (Fig 5); of course, the convergence is now faster. With *w* = 1.0(Fig 3) one needs 99iterations to achieve a convergence limit *at e =0.000*1V. With *w* = 1.2(Fig 4) it is still 67 iterations, and with 1.4(Fig 5) only 40.

With w = 1.6(Fig 7) the iteration converges still faster; now only 18 steps are needed. The series of larger changes, however, no longer resembles a geometric series. The changes start to jump about. With still larger values of w the jumps become stronger, and the convergence is poorer. With w = l.8(Fig 6) one needs 58 iterations, and with w =1.9(Fig 9)one needs 128. With w = 1.99(Fig 8)the jumps dominate. After 500 iterations the changes are. smaller than 1V. The convergence limit is reached after 1339 iterations.

For other values of m and n one finds a similar behavior. If one wishes to specify a standard value for w, this should lie at about w = 1.5 or 1.6.

The standard value for Φ_0 is 1000 V, The method also converges with other values such as, for example, with Φ_0 = 0 V, It then costs a few preliminary iterations, before the approx-Imation achieves the quality which it had from the start with potential Φ_0 =1000V. From then on the behavior of the iterations little different

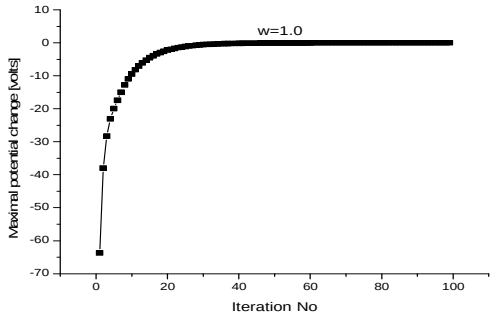

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Fig.3

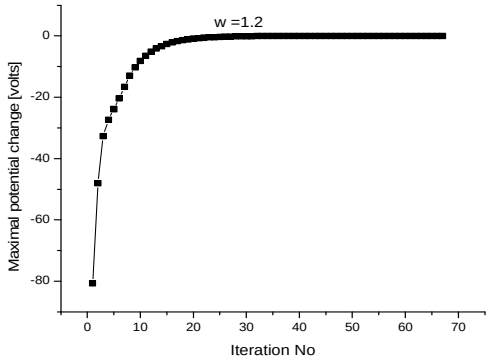

Fig.4

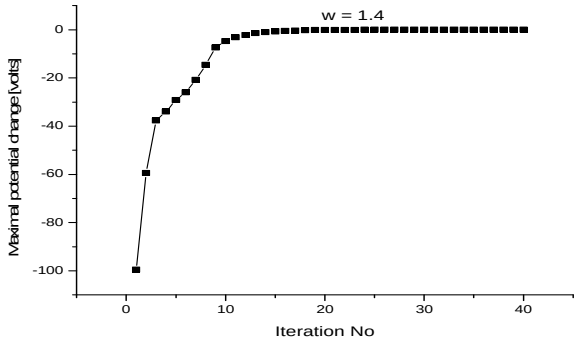

Fig.5

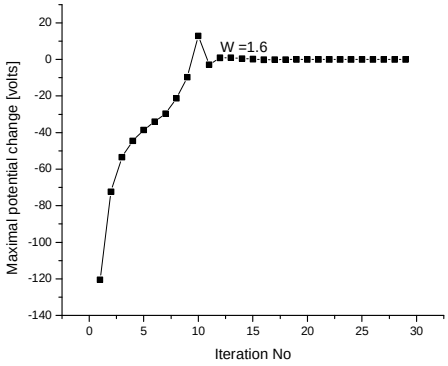

Fig.6

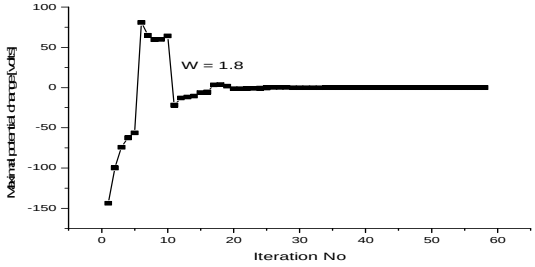

Fig. 7

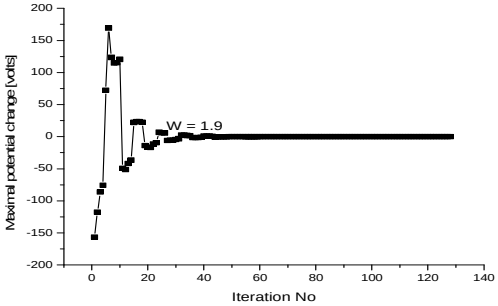

Fig. 8

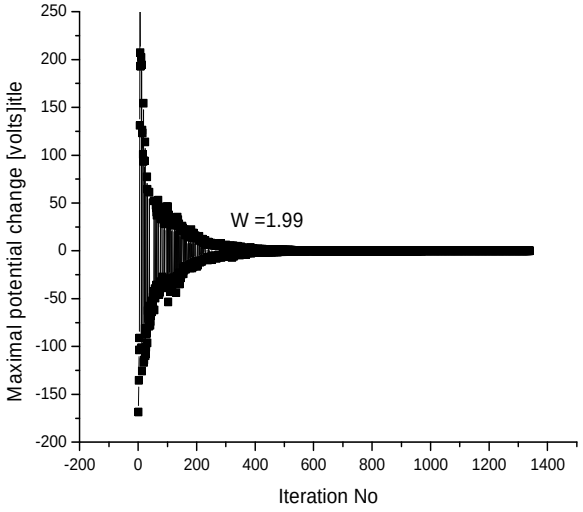

Fig. 9

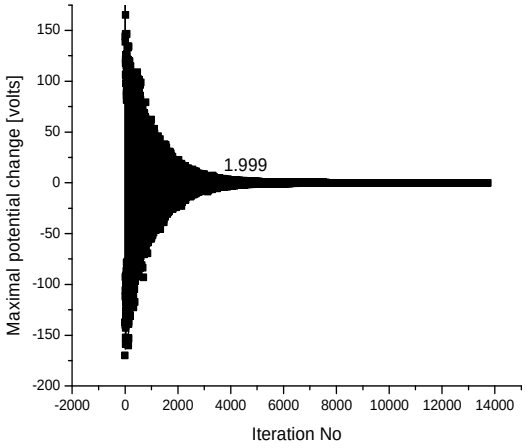

Fig. 10

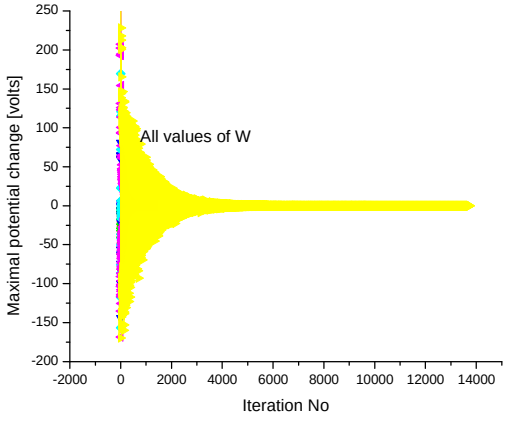

Fig. 11

Conclusion

We have seen so far with w = 1.0 there was slow convergence. The largest changeof the potential grid decreases momentarily from iteration to iteration. However, the behavior was found with w = 1.2 and w = 1.4 but the convergences was faster. It was observed that with w = 1.0 99iterations were needed to achieved convergence limit at e = 0.0001V. with w = 1.2 it was 67iterations and with 1.4 it is 40iterations. The series of larger changes, however, no longer resembles a geometric series, with larger values of w, the jumpsbecome stronger and convergence become poorer. For other values of m and n we found a similar behavior. The standard value for w lied between 1.5 to 1.6 and standard value for Φ_0 was 1000V

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